Solutions to Homework 1

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1. Prove that for every concave function ψ, $\int \psi(\phi(x))\,dx \leq \psi(\int \phi(x)\,dx)$. You are allowed to assume that both ψ and ϕ are continuous bounded functions, and therefore you can use the definition of Riemann integral.

Solution: Definition of a concave function $\psi(a\lambda_1 + b\lambda_2) \geq \lambda_1\psi(a) + \lambda_2\psi(b)$ where $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$.

Claim: $\psi(\sum_{i=1}^{n} \phi(x_i)\lambda_i) \geq \sum_{i=1}^{n} \lambda_i\psi(\phi(x_i))$ where $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$

Proof By induction:

Base case n = 2 holds by definition. Lets assume it to be true for n = k and prove it for n = k + 1.
Lets define $p_i = \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i}$.$p_i \geq 0$ and $\sum_{i=1}^{k} p_i = 1$.
Lets define $S_{k+1} = \sum_{i=1}^{k} p_i\phi(x_i)$
By assumption for n = k we have

$$\psi(\sum_{i=1}^{k} p_i\phi(x_i)) \geq \sum_{i=1}^{k} p_i\psi(\phi(x_i)) \tag{1}$$

Using the base case we have

$$\psi(\lambda_{k+1}\phi(x_{k+1}) + (1 - \lambda_{k+1})S_{k+1}) \geq \lambda_{k+1}\psi(\phi(x_{k+1})) + (1 - \lambda_{k+1})\psi(S_{k+1})$$

from equation (1) we have

$$\begin{aligned} & \psi(\sum_{i=1}^{k+1} \lambda_i \phi(x_i)) \\ \geq \ & \lambda_{k+1}\psi(\phi(x_{k+1})) + (1-\lambda_{k+1})(\sum_{i=1}^{k} p_i \psi(\phi(x_i))) \\ = \ & \lambda_{k+1}\psi(\phi(x_{k+1})) + \sum_{i=1}^{k} \lambda_i \psi(\phi(x_i)) \\ = \ & \sum_{i=1}^{k+1} \lambda_i \psi(\phi(x_i)) \end{aligned}$$

To extend it for continuous case we notice that *Reimann Integral* is *countable* summation. i.e

$$\int \phi(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \phi(x_i)\Delta x_i$$

Assuming the measure on the space is 1 i.e $\int dx = 1$ we have

$$\int \psi(\phi(x))\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \psi(\phi(x))\Delta x_i \leq \psi(\lim_{n\to\infty} \sum_{i=1}^{n} \phi(x_i)\Delta x_i) = \psi(\int \phi(x)\, dx)$$

Q.E.D

2. Prove that any bounded monotonic non-decreasing sequence of real numbers has a limit. In other words for any sequence $x_1, x_2, ..., x_n, x_{n+1}, ...$ such that $\forall i,\ x_i \in \mathcal{R}$ and $\forall i,\ x_{i+1} \geq x_i$ and $\exists M$ such that $\forall i,\ x_i < M \lim_{i\to\infty} x_i$ exists.

Solution: Let M be the *least* upper bound. Then for any $\epsilon > 0\ \exists\ N$ such that $M - \epsilon < x_N \leq M$. If not then $M - \epsilon$ will be the least upperbound. Since the sequence is monotonically non-decreasing $\forall n \geq N, M - \epsilon < x_n \leq M$. In other words the sequencing is converging towards M and hence M is the limit.

Q.E.D

3. Prove that if $X_1, X_2, ..., X_n$ are random variables that $\lim_n supX_n$ is a random variable.

Solution: Let $X_0 = \lim_n sup\ X_n$. Let $A_n = \{\omega : X_n(\omega) \leq t\}$ for some $t \in \mathcal{R}$. Given for $\forall n \geq 1,\ A_n \in \mathcal{F}$ and to prove $A_0 \in \mathcal{F}$. Let $Y_k = \sup_{n\geq k} X_n$. Let $B_k = \{\omega : Y_k(\omega) \leq t\}$. It is clear that $B_k = \cap_{n\geq k} A_n$. We also notice that $X_0 = \inf_{k\geq 1} Y_k$ and hence $A_0 = \cup_{k\geq 1} B_k$.

Hence $A_0 = \cup_{k\geq 1} \cap_{n\geq k} A_n$. $\mathcal{F}$ is closed under countable uniouns and intersections. Hence $A_0 \in \mathcal{F}$.

Q.E.D

4. Consider the function $f(x)$ defined as $f(x) = 1$ if x is rational and $f(x) = 0$ otherwise. Compute the Lesbegue integral $\int_0^1 f(x)\,dx$.

Solution: Many students proved by two limiting arguments where they proved that the $\sigma - algebra$ includes singleton sets and the measure on them is zero and then the countable infinite sum of zero is zero to claim the fact that measure on the set of rational numbers (which is a countably infinite union of disconnected singleton sets) is zero . Using two limiting arguments to prove a theorm is not a sound proof. It is essential to bear in mind that a limiting argument is an infinite process. One cannot *stop* and claim that they have hit upon the set that contains a single rational number(which can never be reached) and *then* have a countable infinite union(through another limiting argument) to produce the set of rational numbers. A better solution is to to merge both these limiting arguments into one.

Proof:

It is clear that if the measure on a set (A) is zero, the measure on any of its subsets is also zero. The set of rational numbers between 0 and 1 is a subset of the set of rational numbers. So it is just enough to prove that the measure on the set of rational numbers is zero. Lets index the rational numbers as $r_1, r_2, ..., r_n, r_{n+1}, ...$ (this is possible as they are countable). Define a set $A_n = (r_n - \frac{\epsilon}{2^n}, r_n + \frac{\epsilon}{2^n}]$ for any $\epsilon > 0$. The width of the set A_n equals $\frac{\epsilon}{2^{n-1}}$ and hence $P(A_n) = \frac{\epsilon}{2^{n-1}}$. Using Boole's inequality we have

$$P(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n) = 2\epsilon$$

Now as $\epsilon \rightarrow 0$ we get measure on the set of rational numbers to be zero.

Q.E.D