

Outline

- Euclidean space \mathbb{R}^3
 - distance, angles, rotations
- Metric Space
 - Distance, angles, rotations
- Least Squares
- Singular Value Decomposition
- Other Matrix decompositions

Motivation

- Fundamental to representation and numerical solution of almost all problems including those in vision and computational statistics.
 - Solving equations for calibration, stereo, tracking, ...
- Geometry is fundamental to vision. However one way of doing geometry is via algebra.
 - Intersections of lines, points, planes. Determining angles. Determining orthogonal projections ...
- Modern computer vision is formulated in terms of “projective geometry”. Most results in projective geometry are stated algebraically and require knowledge of concepts such as rank, null space, constraints

Applications

- Rectification of images
- Calibrating cameras
- Transforming color spaces
- Tracking motion of a rigid body
- Applying constraints from multiple views
- Parametrizing fundamental matrix and trifocal tensor.

Vectors

- A vector \mathbf{x} of dimension d represents a point in a d dimensional space
- Examples
 - A point in 3D Euclidean space $[x,y,z]$ or 2D image space $[u,v]$
 - A point in a projective space $P^3 [X,Y,Z,W]$ or in projective space $P^2 [U,V,W]$
 - Point in color space $[r,g,b]$ or $[y, u, v]$
 - Point in an infinite dimensional functional space on a Fourier basis
 - Vector of intrinsic parameters for a camera (focal length, skew ratio, ...)
- Essentially a short-hand notation to denote a grouping of points
 - No special structure yet

Vectors and Matrices

- d dimensional column vector \mathbf{x} and its transpose

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \quad \text{and} \quad \mathbf{x}^t = (x_1 \ x_2 \ \dots \ x_d)$$

- $d \times n$ dimensional matrix \mathbf{M} and its transpose \mathbf{M}^t

$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & \dots & m_{1d} \\ m_{21} & m_{22} & m_{23} & \dots & m_{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & m_{n3} & \dots & m_{nd} \end{pmatrix} \text{ and}$$

- Transpose indicated with a superscript t or a prime $'$

$$\mathbf{M}^t = \begin{pmatrix} m_{11} & m_{21} & \dots & m_{n1} \\ m_{12} & m_{22} & \dots & m_{n2} \\ m_{13} & m_{23} & \dots & m_{n3} \\ \vdots & \vdots & \ddots & \vdots \\ m_{1d} & m_{2d} & \dots & m_{nd} \end{pmatrix} .$$

Determinant

- Determinant of a 2x2 matrix $m_{11}m_{22}-m_{12}m_{21}$
- For a higher dimensional matrix we have a recursive definition

$$i \begin{pmatrix} m_{11} & m_{12} & \cdots & \overset{j}{\otimes} & \cdots & \cdots & m_{1d} \\ m_{21} & m_{22} & \cdots & \otimes & \cdots & \cdots & m_{2d} \\ \vdots & \vdots & \ddots & \otimes & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \otimes & \cdots & \cdots & \vdots \\ \otimes & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\ \vdots & \vdots & \cdots & \otimes & \cdots & \ddots & \vdots \\ m_{d1} & m_{d2} & \cdots & \otimes & \cdots & \cdots & m_{dd} \end{pmatrix} = \mathbf{M}_{i|j}. \quad (22)$$

Given the determinants $|\mathbf{M}_{x|1}|$, we can now compute the determinant of \mathbf{M} the expansion by minors on the first column giving

$$|\mathbf{M}| = m_{11}|\mathbf{M}_{1|1}| - m_{21}|\mathbf{M}_{2|1}| + m_{31}|\mathbf{M}_{3|1}| - \cdots \pm m_{d1}|\mathbf{M}_{d|1}|, \quad (23)$$

where the signs alternate. This process can be applied recursively to the successive (smaller) matrixes in Eq. 23.

Only for a 3×3 matrix, this determinant calculation can be represented by “sweeping” the matrix, i.e., taking the sum of the products of matrix terms along a diagonal, where products from upper-left to lower-right are added with a positive sign, and those from the lower-left to upper-right with a minus sign. That is,

$$\begin{aligned} |\mathbf{M}| &= \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} \\ &= m_{11}m_{22}m_{33} + m_{13}m_{21}m_{32} + m_{12}m_{23}m_{31} \\ &\quad - m_{13}m_{22}m_{31} - m_{11}m_{23}m_{32} - m_{12}m_{21}m_{33}. \end{aligned} \quad (24)$$

Determinant: Remarks

- Determinant determines “magnitude” of matrix. Matrix with determinant =0 is called singular.
- Determinant is important in theorems
- Practically the way to compute the determinant is not this way.
- Homework problem -- determine number of operations for recursive algorithm.

Matrix basics

- Square matrix: number of rows = number of columns
- Symmetric matrix $A_{ij}=A_{ji}$.
- Skew symmetric matrix $A_{ij}=-A_{ji}$.
- Identity $\mathbf{I}_{ij}=\delta_{ij}$
 - Kronecker delta $\delta_{ij}=0$ if $i\neq j$ $\delta_{ij}=1$ if $i=j$
- Lower triangular
- Upper triangular

$$\begin{bmatrix} a & 0 & \dots & 0 \\ c & b & \ddots & \vdots \\ \vdots & \dots & \ddots & 0 \\ f & g & \dots & d \end{bmatrix}$$

$$\begin{bmatrix} a & c & \dots & d \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & e \\ 0 & \dots & 0 & d \end{bmatrix}$$

Matrix vector product

- $m \times n$ dimensional matrix \mathbf{M} multiplies by a n dimensional vector \mathbf{x} to produce a m dimensional vector

$$\begin{pmatrix} m_{11} & m_{12} & \dots & m_{1d} \\ m_{21} & m_{22} & \dots & m_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nd} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

$$y_j = \sum_{i=1}^d m_{ji} x_i.$$

Linear systems of equations

- Systems of equations
- Can be written as a matrix vector product

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

- Can change or scale rows

$$\mathbf{Ax} = \mathbf{b}$$

- Solved via Gauss elimination

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Reduce system to product of

lower or upper triangular matrix and \mathbf{x}

- $O(N)$ operations to solve triangular system
- $O(N^3)$ operations to perform Gauss elimination

LU decomposition

- Any matrix can be written as a product of a lower triangular and upper triangular matrix.
- Most used algorithm in linear algebra.

$$\mathbf{A}=\mathbf{LU}$$

- Practically implemented by reordering equations and scaling them so that loss of accuracy is minimized.

$$\mathbf{A}=\mathbf{PLU}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Scaled LU decomposition with “partial pivoting”
- When \mathbf{A} is symmetric positive definite $x^t Ax > 0$ for all x
“Cholesky decomposition”

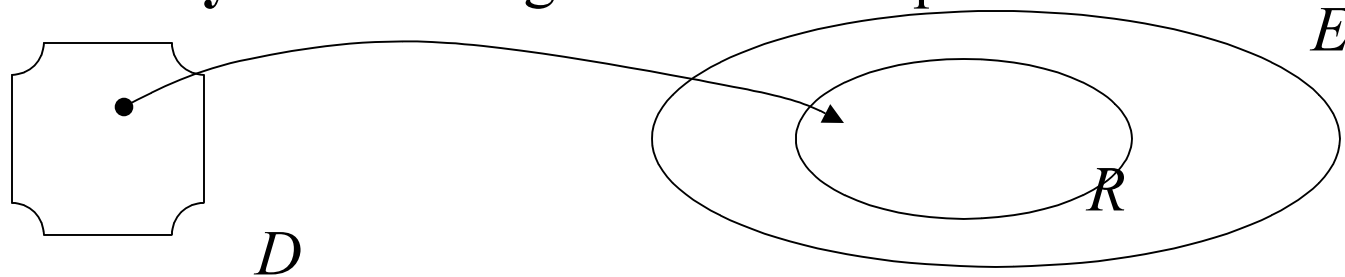
$$\mathbf{A}=\mathbf{LL}^t$$

Linear/Vector Spaces

- Previous stuff was somewhat mechanical.
- In vision we have to answer questions when
 - Models provide equations that are singular or degenerate. What can we say about the solutions? Can we restrict them?
 - Number of unknowns may be more or less than the number of observations. Can we still obtain a meaningful solution?
 - How “far” is an approximation from a solution? How do we measure this distance?
 - Matrices are operators that take one vector into another. What can we say about the properties of the operator? When is an equation involving an operator solvable?

Operators

- Function, Transformation, Operator, Mapping: synonyms
- A function takes elements \mathbf{x} defined *on* its “Domain” D to elements \mathbf{y} in its “Range” R which is part of E



- If for each \mathbf{y} in R there is exactly one \mathbf{x} in D the function is one-to-one. In this case an inverse exists whose domain is R and whose range is D
- We are interested in situations where R and D are finite-dimensional linear spaces

Vector Space

- A collection of points that obey certain rules

– Commutative, existence of a zero element
 $u + v = v + u$; $u + (v + w) = (u + v) + w$

$\exists 0$, $u + 0 = u \quad \forall u$; $u + (-u) = 0$

– Scalar multiplication

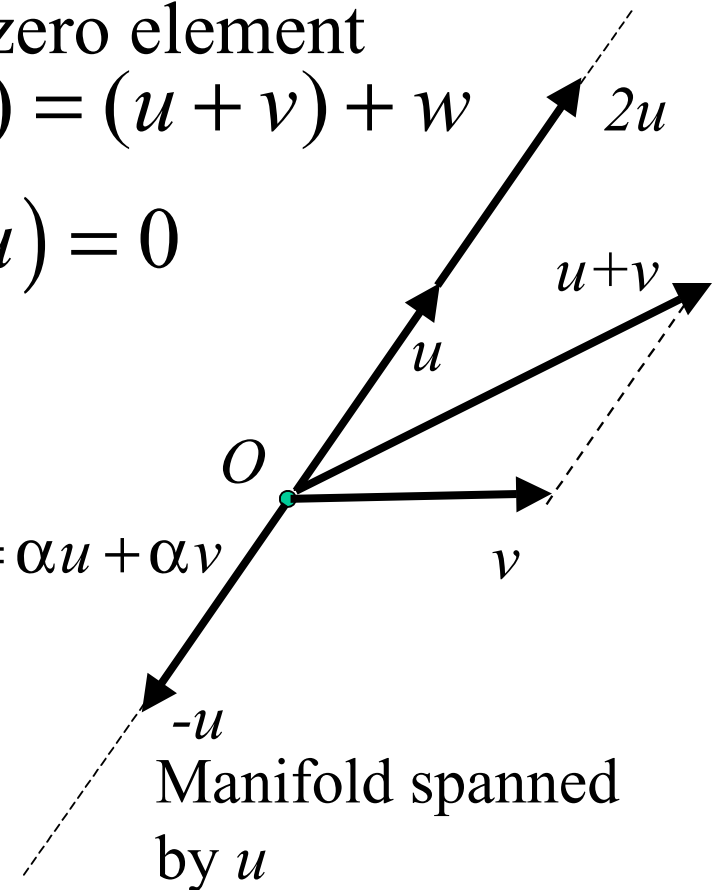
$\alpha(\beta u) = (\alpha\beta)u$; $1u = u$

$(\alpha + \beta)u = \alpha u + \beta u$; $\alpha(u + v) = \alpha u + \alpha v$

- Let $\mathbf{u}_1, \dots, \mathbf{u}_k$ be a set of vectors:

Linear combination

$$\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k$$



Dependence and dimensionality

- A set of vectors is dependent if for some scalars $\alpha_1, \dots, \alpha_k$ not all zero we can write
$$\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k = \mathbf{0}$$
- Otherwise the vectors are independent.
- If the zero vector is part of a set of vectors that set is dependent. If a set of vectors is dependent so is any larger set which contains it.
- A linear space is n dimensional if it possesses a set of n independent vectors but every $n+1$ dimensional set is dependent.
- A set of vectors $\mathbf{b}_1, \dots, \mathbf{b}_k$ is a basis for a k dimensional space X if each vector in X can be expressed in one and only one way as a linear combination of $\mathbf{b}_1, \dots, \mathbf{b}_k$
- One example of a basis are the vectors $(1,0,\dots,0), (0,1,\dots,0), \dots, (0,0, \dots, 1)$

Distances/Metrics and Norms

- We would like to measure distances and directions in the vector space the same way that we do it in Euclidean 3D
- Distance function $d(\mathbf{u}, \mathbf{v})$ makes a vector space a metric space if it satisfies
 - $d(\mathbf{u}, \mathbf{v}) > 0$ for \mathbf{u}, \mathbf{v} different
 - $d(\mathbf{u}, \mathbf{u}) = 0$, $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
 - $d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$ (*triangle inequality*)
- Norm (“length”).
 - $\|\mathbf{u}\| > 0$ for \mathbf{u} not 0, $\|\mathbf{0}\| = 0$
 - $\|\alpha \mathbf{u}\| = |\alpha| \|\mathbf{u}\|$, $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$
- Normed linear space is a metric space with the metric defined by $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$ and $\|\mathbf{u}\| = d(\mathbf{u}, \mathbf{0})$

Dot Product

- Dot product of two vectors with same dimension

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^t \mathbf{y} = \sum_{i=1} x_i y_i = \mathbf{y}^t \mathbf{x}.$$

- Dot product space behaves like Euclidean \mathbb{R}^3
- Dot product defines a norm and a metric.

- Parallelogram law

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

- Orthogonal vectors $\langle \mathbf{u}, \mathbf{v} \rangle = 0$
- Angle between vectors

$$\cos \theta = \langle \mathbf{x}, \mathbf{y} \rangle / \|\mathbf{x}\| \|\mathbf{y}\|$$

- Orthonormal basis -- elements have norm 1 and are perpendicular to each other
- Other distances and products can also define a space:
 - Mahalanobis distance

Matrices as operators

- Matrix is an operator that takes a vector to another vector.
 - Square matrix takes it to a vector in the space of the same dimension.
- Dot product provides a tool to examine matrix properties
 - Adjoint matrix $\langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{A}^* \mathbf{v} \rangle$
 - Square Matrix fully defined as result of its operation on members of a basis.

$$A_{ij} = \langle \mathbf{A}\mathbf{b}_j, \mathbf{b}_i \rangle$$

Eigenvalues and Eigenvectors

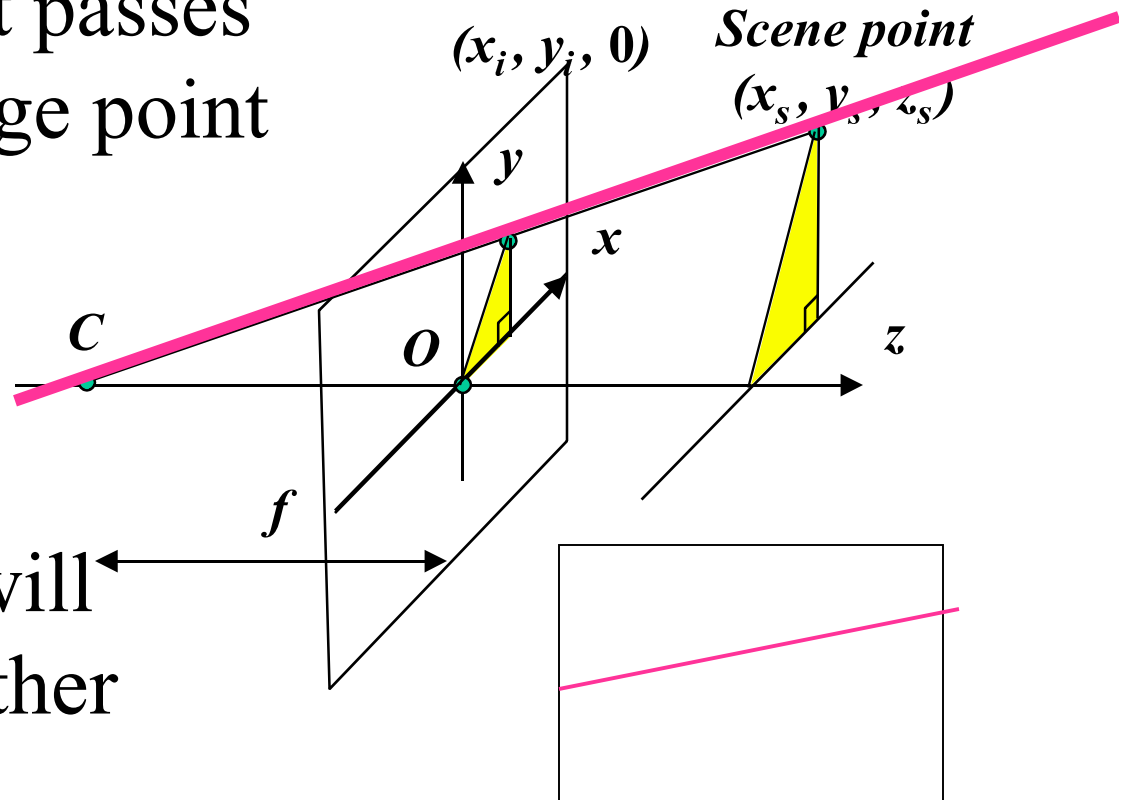
- Square matrix possesses its own natural basis.
- Eigen relation

$$\mathbf{A}\mathbf{u}=\lambda\mathbf{u}$$

- Matrix \mathbf{A} acts on vector \mathbf{u} and produces a scaled version of the vector.
- Eigen is a German word meaning “proper” or “specific”
- \mathbf{u} is the eigenvector while λ is the eigenvalue.
 - If \mathbf{u} is an eigenvector so is $\alpha\mathbf{u}$
 - If $\|\mathbf{u}\|=1$ then we call it a normal eigenvector
 - λ is like a measure of the “strength” of \mathbf{A} in the direction of \mathbf{u}
- Set of all eigenvalues and eigenvectors of \mathbf{A} is called the “spectrum of \mathbf{A} ”

Motivation: Stereo

- Point (x, y) on the image plane lies on a line in the world that passes through the image point and center of projection



- Image of this line in the world will form a line in another camera

Epipolar Constraint

- Point in one image lies on the “epipolar line” in the other image
- Algebraic statement of geometry
 - Equation of line in the other image is \mathbf{Fm}
 - Condition that the point \mathbf{m}' lies on this line is
$$\mathbf{m}' \cdot \mathbf{Fm} = 0$$
- \mathbf{F} is the “fundamental matrix”
- Estimating the fundamental matrix is an important problem in vision

$$\sum m_i m_j F_{ij} = 0$$

Eight point algorithm: Determining the Fundamental matrix

- Given a set of matching points in the images, Determine **F**

$$[u' \quad v' \quad 1] \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$
$$u'(f_{11}u + f_{12}v + f_{13}) + v'(f_{21}u + f_{22}v + f_{23}) + (f_{31} + f_{32} + 1) = 0$$

Determining F

- Write expression as an equation in the unknown elements.
- If we have eight points we can solve for elements of F , (e.g. via LU)

$$\begin{bmatrix} u'u & u'v & u' & v'u & v'v & v' & 1 & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = [-1]$$

- **If we have more than eight points we can use a least squares formulation**