

# CAP5416

## Homework 1

This assignment is due on Tuesday, January 30, 2007. If you have any difficulty completing the assignment before the deadline, please inform the instructor or the TA promptly. The first two problems are problems from linear algebra, and the remaining four problems are from the textbook by B. Horn.

### Problem 1      Orthogonal Iteration

Eigenvalues and eigenvectors are perhaps the most important linear algebraic concepts. In this problem, we provide a simple exercise on demonstrating how the eigenvalues and eigenvectors can be computed in general using an algorithm called QR iteration. For this problem, you need to use MATLAB.

Let's start with three vectors,  $a = [1 \ -1 \ 3]^t$ ,  $b = [-1 \ -1 \ 0]^t$  and  $c = [3 \ -3 \ -2]^t$ .

**A.** Show that these three vectors are linear independent. (You can use any method you like to prove this. However, write out your argument clearly).

**B.** Let  $X$  be the matrix whose columns are these three vectors :

$$X = \begin{bmatrix} 1 & -1 & 3 \\ -1 & -1 & -3 \\ 3 & 0 & -2 \end{bmatrix},$$

and  $D$  be the following diagonal matrix

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, let  $A = XDX^{-1}$ . Is the matrix  $A$  symmetric?. What are the eigenvalues and eigenvectors of  $A$ ? Except computing  $A$ , you should not calculate anything for these two questions.

**C.** The orthogonal iteration for computing eigenvalues and eigenvectors of  $A$  is shown in the figure below. According to this algorithm,  $\hat{Q}_k$  will converge (for sufficiently large  $k$ ) to some orthogonal matrix  $\hat{Q}$ , and  $R_k$  will also converge (for our particular matrix  $A$ ) to an upper triangular matrix  $R$ . The eigenvalues of  $A$  arranged in descending order are the diagonal entries of  $R$ . The eigenvectors of  $A$  can be computed from entries of  $R$  and columns of  $Q$ .

Let  $X_0$  be the following matrix

$$X_0 = \begin{bmatrix} 0.9501 & 0.4860 & 0.4565 \\ 0.2311 & 0.8913 & 0.0185 \\ 0.6068 & 0.7621 & 0.8214 \end{bmatrix}.$$

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## Orthogonal Iteration

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$\mathbf{X}_0 =$  arbitrary  $n \times p$  matrix of rank  $p$ .

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for  $k = 1, 2, \dots$  do  
     $\hat{Q}_k R_k = \mathbf{X}_{k-1}$     Reduced QR factorization of  $\mathbf{X}_{k-1}$   
     $\mathbf{X}_k = A \hat{Q}_k$   
end for
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Show the upper triangular matrices  $R_1, R_2, R_3, R_4, R_6$ . Write out all these matrices for me to see. For this part, you need to use MATLAB. (The MATLAB command 'qr' does the QR factorization and to compute  $X^{-1}$ , use 'inv(X)'). You do not need to show the MATLAB code you use. However, the code should not be longer than 7-8 lines. Of course, you are allowed to compute  $R_7, R_8, \dots$  for your own enlightenment.

**D.** From Part C, you can see that the diagonal entries of  $R_6$  are good approximations of the true eigenvalues. How about the columns of  $\hat{Q}_6$ ? Are they good approximations of the true eigenvectors?

## Problem 2 Symmetric Matrices

For this problem, we will prove an important fact about the eigenvalues and eigenvectors of symmetric matrices. That is, all eigenvalues of an  $n \times n$  symmetric matrix  $A$  are real, and furthermore, the eigenvectors of  $A$  form an orthonormal basis of  $\mathbb{R}^n$ . An example of this theorem is given in the previous problem. It is difficult to overstate the importance of this result for computer vision, computer graphics or any other applied science in general.

Recall that for any pair of (real) vectors  $v$  and  $w$ , we can compute their inner product (or dot-product) denoted by  $\langle v, w \rangle$ :

$$\langle v, w \rangle = v^t w.$$

The inner product has one nice property that the inner product of a non-zero vector with itself is a positive real number. And, its square root  $\sqrt{\langle v, v \rangle}$  is the "length" of the vector. Furthermore, if  $\lambda$  is any real number,

$$\langle v, \lambda w \rangle = \lambda \langle v, w \rangle = \langle \lambda v, w \rangle.$$

Now, let  $A$  be a matrix. From the identities below:

$$v^t(Aw) = v^tAw = (v^tA)w = (A^tv)^tw$$

we have

$$\langle v, Aw \rangle = \langle A^tv, w \rangle .$$

Therefore, if  $A$  is symmetric, meaning  $A^t = A$ , we have

$$\langle v, Aw \rangle = \langle Av, w \rangle .$$

Remember, this works for real matrices and vectors. What about complex vectors and matrices? For example, consider the following complex vector

$$v = \begin{bmatrix} i \\ i \end{bmatrix} .$$

If you compute its inner product with itself like above, i.e. take the sum of the squared components, you end up with  $-2$ . That is, you cannot find a positive real number to define the 'length' of this vector. However, if you compute the inner product differently, using the conjugates of the components, then, there will be no such problem. This is what is called Hermitian inner product.

Let  $v$  and  $w$  be two complex vectors, i.e. components of  $v$  and  $w$  are complex numbers. Let  $v = (v_1, \dots, v_n)$  and likewise  $w = (w_1, \dots, w_n)$ . The Hermitian inner product of  $v$  and  $w$  is defined as

$$\langle v, w \rangle_H = \sum_{i=1}^n \bar{v}_i w_i .$$

That is, we do not take the sum of products of components as before. Instead, we modify it a little bit so that the products are between the conjugates of the components of the vector  $v$  and the component of  $w$ <sup>1</sup>. In vector notation, this is

$$\langle v, w \rangle_H = \bar{v}^t w .$$

In the above,  $\bar{v}$  is a vector whose components are conjugates of components of  $v$ . For example, if  $v = [2 + 3i \ 3 - 4i]^t$ ,  $\bar{v} = [2 - 3i \ 3 + 4i]^t$

**A.** Show that the Hermitian inner product of the vector  $v = [i \ i]^t$  above is 2.

**B.** Show that if  $\lambda$  is any complex number, then

$$\langle v, \lambda w \rangle_H = \lambda \langle v, w \rangle_H . \tag{1}$$

$$\langle \lambda v, w \rangle_H = \bar{\lambda} \langle v, w \rangle_H . \tag{2}$$

**C.** Show that if  $v$  and  $w$  are real vectors, their Hermitian inner product is the same as the usual inner product.

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<sup>1</sup>For a complex number  $a + ib$ , its conjugate is  $a - ib$ .

**D.** Show that if  $A$  is a real symmetric matrix,

$$\langle v, Aw \rangle_H = \langle Av, w \rangle_H .$$

**E.** Using Part D and Equations 1 and 2, show that if  $\lambda$  is an eigenvalue of a symmetric matrix  $A$ , then  $\lambda$  is real. (The argument is very short .... Take  $v$  to be an eigenvector.)

In the following  $A$  will denote a symmetric matrix. We now show that eigenvectors of  $A$  form an orthonormal basis.

**F.** Let  $v$  be an eigenvector of  $A$ . We know from Part E that  $v$  is a real vector (because it has real eigenvalues). Let  $\mathcal{S}$  be the subspace orthogonal to  $v$ . That is, every vector  $x \in \mathcal{S}$  is perpendicular to  $v$ :

$$\mathcal{S} = \{x \mid \langle x, v \rangle = 0\}.$$

Show that  $\mathcal{S}$  is an invariant space, i.e. if  $x \in \mathcal{S}$ ,  $Ax$  is also in  $\mathcal{S}$ . (Hint: The argument should be short.)

The proof is then simple after we have reached this point. If  $v_1$  is an eigenvector of  $A$ , then, from Part F above,  $A$  is a symmetric matrix on the subspace  $\mathcal{S}$  orthogonal to  $v$ . Then, from Parts E and F, we can find another eigenvector of  $A$ , say,  $v_2$  in  $\mathcal{S}$ , and  $v_2$  is perpendicular to  $v_1$ . Now, we look at the subspace  $\mathcal{S}_2$  which is orthogonal to both  $v_1$  and  $v_2$ . Part F shows that  $A$  will again be a symmetric matrix on  $\mathcal{S}_2$ , and so on. At the end, we get a sequence of orthonormal eigenvectors of  $A$ ,  $v_1, v_2, \dots, v_n$ , which finishes the proof.

## Problems 3-6

Problems 2.1, 2.2, 2.3 and 2.12 in the textbook *Robot Vision*.