Recent advances in Complexity CIS 6930/CIS 4930	October 10, 2002

Lecture 14

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1 Introduction

Today we finished the lower bound proof for monotone functions.

Lemma 1. *For every monotone circuit C, the number of negative test graphs for which $C \geq \hat{C}$ does not hold is at most size$(C) \cdot m^2 \cdot \left(\frac{\binom{l}{2}}{k-1}\right)^p \cdot (k-1)^n$*

Proof. Look at the approximations of $\vee$ and $\wedge$ gates and show that at each stage of the construction of $\hat{C}$ fails. I.e. it gives another output than the original output of $\vee$ or $\wedge$ in C in at most $m^2 \cdot \left(\frac{\binom{l}{2}}{k-1}\right)^p \cdot (k-1)^n$ negative test graphs.

We will start by looking at $\vee$ gates. Consider the $\vee$ gate with two inputs: $A = \vee_{i=1}^{r} \lceil X_i \rceil$ and $B = \vee_{i=1}^{s} \lceil Y_i \rceil$. From the previous lecture we know that the new approximator is obtained by performing ar most $2m$ pluckings on $A \vee B$. Now we are going to look at how many negative test graphs are destroyed by each plucking. More specifically we will look at the following:

Let $X_1, ..., X_p$ be the petals of a sunflower with center Z. What is the probability that $\lceil Z \rceil$ accepts a negative test graph, but none of the $X_1, ..., X_p$ accept the same negative test graph.

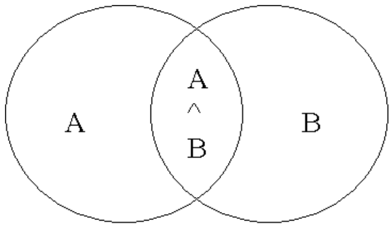
This will only happen if and only iff the vertices of Z are assigned distinct colors (DC), but every petal X_i has two vertices with the same color. We now get the following

$$Prob[Z \text{ is } DC \text{ and } X_1, ..., X_p \text{ are not DC}] \leq Prob[X_1, ..., X_p \text{are DC } | Z \text{ is DC}] \tag{1}$$

$$= \prod_{i=1}^{p} Prob[X_i \text{ is not DC } | Z \text{ is DC}] \tag{2}$$

$$\leq \prod_{i=1}^{p} Prob[X_1, ..., X_p \text{ is not DC }] \tag{3}$$

Now the first one is due to the definition of conditional probability, see figure 1. The second inequality holds due to the fact that all the individual events



If we would ask what the chance is that A would occur given that B has occured, we can see that it is similar to asking what the chance is that $A \wedge B$ occur.

Figure 1: $Prob(A \wedge B) = Prob(A|B)$

$Prob[X_i$ is not DC $|Z$ is DC$]$ are independent of each other. The last inequality is left as an exercise. Now we have that $Prob[X_i$ is not DC $] \leq \frac{\binom{l}{2}}{k-1}$. This follows from the the fact that the probability that a fixed pair has the same color is $\frac{1}{k-1}$ and there are $\binom{l}{2}$ of these pairs. Now since we have p of these inequalities we have:

$$Prob[Z \text{ is DC and } X_1, ..., X_p \text{ are not DC}] \leq \left(\frac{\binom{l}{2}}{k-1}\right)^p.$$

Plucking adds at most $(\frac{\binom{l}{2}}{k-1})^p$ new graphs. There are at most $2m$ pluckings, so the total of number of negative graphs violating the inequality is $2m \cdot (\frac{\binom{l}{2}}{k-1})^p$.

For the $\wedge$ gates we have a similar reasoning with the observation that the *replacement* and *throwing away* stages do not introduce new violations. However we do have m^2 pluckings instead of $2m$. □

Exercise

Exercise 1. *Show $Prob(A|B)$ is at most $Prob(A)$ under the condition that B is negatively correlated with A, i.e, $\neg B \rightarrow A$. Discuss also why this condition is true in the particular situation of plucking the the clique indicator X_i's to get a clique indicator Z ; here A is the event that X_i is Not distinct colored and Z is the event that Z Is distinct colored?*

Exercise

Exercise 2. *Look at proof in Book which shows that:*

Task 1: *Construction of small l,m,p-approximator for a small monotone circuit.*

Lemma 1: *Small l,m,p-approximators do not do a good job on positive and negative test graphs for clique$_{n,k}$*

Lemma 2a: *l,m,p approximators do atleast as well as their corresponding monotone circuit on most positive test graphs*

Lemma 2b: *l,m,p approximators do at least as well as their corresponding monotone circuit on most negative test graphs*

together imply the Theorem that clique$_{n,k}$ requires monotone circuits of size at least $n^{\Omega(\sqrt{k})}$ for k at most $n^{\frac{1}{4}}$.

Does one need the condition that k is at most $n^{\frac{1}{4}}$? Are there any obvious improvements possible, if not, why not? Are there any alternatives possible, i.e, for larger k's, you get a slightly worse (but still exponential) size lower bound?