Recent advances in Complexity CIS 6930/CIS 4930	December 03, 2002
Lecture 28	
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In this lecture we briefly introduce various topics relating physics, information and computation.

1 Logical Depth

Consider a book on number theory. It will usually list a number of difficult theorems. Therefore it may be considered deep. However, most of thetheorems are derivable from the initial few definitions. Therefore one may conclude that it has a low Kolmogorov Complexity. The estimate of depth is based on the fact that it takes a long time to derive the results in the book given the information it contains. In other words a sequence may be considered deep if it yields it secrets slowly. One can derive the regularities in it only using sufficiently lengthy analysis.

Definition 1 *Logical depth is the necessary number of steps in the deductive or causal path connecting an object with its plausible origin. Formally, it is the time required by a universal computer to compute the object from its compressed original description.*

2 Reversible Computation

As circuits shrink smaller, they result in higher bit densities and higher amount of heat per bit. Many people feel that heat will ultimately prove to be the greatest challenge facing the semiconductor industry. This provides a motivation for reducing the heat generated. Additionally, lesser heat will mean lower power consumption and this in turn will mean longer lasting batteries for laptops. The key to all this, some scientists believe is in reversible computing. In this section we look at reversibility primarily at the logical level. We look at some of the reversible logic gates that have been proposed over the years.

2.1 What is Reversibility?

Computers can be thought of as engines that must dissipate energy in order to process information. Landauer throuroughly analyzed the problem of heat dissipation and concluded that it is only logically irreversible operations that must dissipate energy. An operation is logically reversible if its inputs can always be deduced from the outputs. Erasure of information is not reversible. Erasing each bit costs $kT\ ln2$ energy when the computer operates at temperature T.

Landauer reasoned as follows: Distinct physical states of a computer are needed to represent distinct physical states of the computer hardware. Each bit has one degree of freedom and so n bits have n degrees of freedom and they represent the 2^n physical states. What happens when n bits are irreversibly erased? Assume that an erasure amounts to resetting the bits to 0. Before erasure there are 2^n distinct states of the n bits. After erasure there is only one state (all bits are 0). According to the second law of thermodynamics this loss of degrees of freedom of the physical system must be compensated for. This can happen only by heat dissipation. Therefore only irreversible computers need necessarily dissipate heat. A computer that appended the input bits to the result bits was proposed and the nice thing is that the energy required depended not on the complexity of the computation but only on the number of result bits that needed to be reset after the computation.

2.2 Reversible Gates

To start with, consider the conventional $NAND$ gate. Observe that it is not a reversible gate. By this we mean that the input to the gate cannot be constructed given the output. For example if the output is 1 the input may have been (0,0) or (0,1) or (1,0). The AND, OR and XOR gates are all not reversible. The input information is irreversibly lost. We call an operation reversible if it is possible to deduce the input, given the output. An example of a reversible gate is the NOT gate. If the output is 1 the input is 0 and if the output is 0 we can deduce that the input was 1. We present some of the other popular reversible gates.

2.2.1 Controlled NOT (CN) gate

This is a two input two output gate. The NOT operation is operative on the lower line (say b) only, that too b is inverted only when the upper line (a) has an input 1. The upper line output is the same as the upper input. We can interpret the lower line output, b' as being the XOR of the two inputs a and b. By examining the truth table of the CN gate, one can easily verify the fact that it is reversible. Unfortunately it is not possible to perform all possible operations using only the NOT and the CN gates. For this we need another gate called the CCN gate, also known as the Toffoli gate.

2.2.2 Controlled Controlled NOT (CCN) Gate

This gate has two control lines, a and b. As in case of the CN gate the signals on these lines simply pass through. The third line, c is the NOT line. However this is activated only when the first two lines input 1. Observe that with a and b set to 1 this gate functions as a NOT gate and with a set to 1 it is a CN gate. That's not all. In fact the CCN gate can do all possible operations. So it is a universal gate!

2.2.3 Fredkin Gate

Fredkin proposed an additional constraint on the gates. He developed a gate that is not only reversible but also conserves the number of 0s and 1s, i.e., the number of input 0s is same as the number of output 0s and the same is true for 1s. The gate performs a controlled exchange. If input a is 0, the the inputs b

and c are not exchanged. If a is 1 they are exchanged. It can be shown that the Fredkin gate can be used to perform all logical operations.

3 Thermodynamics

Classical thermodynamic deals with the description of physical properties of substances under variations of macroscopic observables such as volume, temperature and preassure. The two fundamental laws of thermodynamics are:

First Law: The total energy of an isolated system is invariant over time.

Second Law: No process is possible that has as its only result the transformation of heat into work.

3.1 Entropy

3.1.1 Classical Entropy

The classical definition of entropy is due to Carnot and he proposed it in the context of the so called Carnot cycle. He defines entropy as a function of temperature and volume as $S(T_b, V_b) = S(T_a, V_a) + \int_a^b (dQ/T)$, where the integral is taken over a reversible path between states a and b. Note that the answer does not depend on the choice of the reversible path. This classical definition determines the entropy S only up tp an additive constsnt. If the process is irreversible then $\oint dS > 0$. Thus, entropy always increases with time, except for irreversible systems, where it can stay the same.

3.1.2 Statistical Mechanics and Boltzmann Entropy

Boltzmann was the first to formulate the idea that entropy measures disorderliness, and to connect this with the origin of the distinction between past and future in a non relativistic universe. To begin with we make the following distinction between macro and micro states. The macro state of an isolated mechanical system consists of an approximate description of just a few macroscopic observables such as volume, temperature pressure, and so on. The micro state of a system consists of an exhaustive description of the values of microscopic parameters of the particles. The goal of statistical mechanics is to derive the classical laws of thermodynamics from microscopic phenomena.

Definition 2 *The Boltzmann entropy of a system with macroscopic description* x *is* $S_B(x) = (kln2)logL(\Gamma_x)$, *where* $k = 1.38 \times 10^{-23}$ *joules/kelvin is called the Boltzmann constant, and* Γ_x *is the cell in state space corresponding to the macroscopic description* x *of the system, and* $L(\Gamma_x)$ *is its volume.*

3.1.3 Algorithmic Entropy

So far macroscopic parameters were not accounted for. Suppose that we determine the macroscopic parameters of a thermodynamical system, we truncate them to the required precision and finally encode them in an integer x.

Definition 3 *Suppose the system in equillibrium is described by the encoding* x *of the approximated macroscopic parameters. The algorithmic entropy* *of the macro state of a system is given by* $K(x) + H_x$*, where* $K(x)$ *is the prefix complexity of* x*, and* $H_x = S_B(x)/(kln2)$*. Here* $S_B(x)$ *is the Boltzmann entropy of the system constrained by macroscopic parameters* x*, and* k *is Boltzmann's constant. The physical version of algorithmic entropy is defined as* $S_A(x) = (kln2)(K(x) + H_x)$*.*

3.2 Maxwell' s Demon

Maxwell' s Demon is a paradox discovered by James Maxwell in 1871. It resulted in controversy among physicists for over a century and was eventually resolved. The pursuit of a satisfactory explanation for the paradox has lead to studies in reversible computing and energy of computing.

3.2.1 The Problem

Before stating the problem recall that according to the second law of thermodynamics, the entropy of an isolated system either remains constant or increases. It cannot decrease spontaneously.
Imagine that we have a box filled with gas at some particular temperature. As we are aware, the temperature determines the average speed of the molecules of the gas. However, some of the gas molecules will be moving faster than average speed and some of the gas molecules will be moving slower than average. Now imagine that a partition is placed across the middle of the box so as to separate the two sides of the box. We now have sides of the box filled with gas at the same temperature. Maxwell imagined a small (about the size of a molecule) trap door in the partition with his demon controlling the door. The demon observes molecules moving towards the door from both directions. He measures the speed of such molecules and makes the following decision. When he sees a faster than average molecule approaching the door he makes certain that it ends up in the left partition by opening the door if it is coming from the right and leaving it closed otherwise. Similarly, when a slower than average molecule approaches the door he makes sure that it ends up in the right partition. Soon, we obtain a box with all the faster than average molecules in the left partition and all slower than average molecules in the right partition. Therefore the box is hot on the right and cold on the left. We have reduced the entropy of the system and this is a violation of the second law of thermodynamics!

3.2.2 What happens to the Entropy?

Since Maxwell came up with the paradox, people have attempted to point the flaw in the argument. Many scientists believed that entropy was generated during the measurement process. For example the demon may need to flash photons at the molecules to observe them and measure their speed and this resulted in entropy increase in the environment. However, Bennet showed that these measurement can actually be made with zero energy expenditure provided certain rules are followed while obtaining and erasing information relating to the measurements. The demon is assumed to be in a standard state (S) before a measurement. When a molecule approaches, it classifies the molecule to be either R (right moving) or L (left moving) and moves to the corresponding state. Later, this information needs to be erased and the state reset to S in preparation for the arrival of the next molecule. It has been shown that this erasure step is one that releases entropy. This is now the generally accepted view and it provides a link between studies in thermodynamics and information theory.