FINITE RESOLUTION CRISP AND FUZZY SPATIAL OBJECTS

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ABSTRACT

Uncertainty management for geometric data is currently an important problem in spatial databases, image databases, and GIS. Spatial objects do not always have homogeneous interiors and sharply defined boundaries but frequently their interiors and boundaries are partially or totally indeterminate and vague. For an important kind of spatial vagueness called *fuzziness* this paper provides a conceptual model of fuzzy spatial objects that also incorporates fuzzy geometric union, intersection, and difference operations as well as fuzzy topological predicates. In particular, this model is not based on Euclidean space and on an infinite-precision arithmetic which lead to lacking numerical robustness and to topological inconsistency; it rests on a finite, discrete geometric domain called *grid partition* which takes into account finite-precision number systems available in computers. Last but not least this paper shall also be a contribution to achieve a uniform treatment of vector and raster data.

1 INTRODUCTION

Currently, *spatial data modeling*, as the basis of spatial databases and GIS, is affected by two main problems. The first problem is that it has so far been suffering from the “boundary syndrom” which relates to the imagination that the extent of a spatial object is or has to be always limited by a precisely defined and abrupt *boundary* separating the interior of the object from its exterior. In practice, however, there is no apparent reason for the whole boundary of a region to be determined. There are many geographical application examples which illustrate that the boundaries of spatial objects (like geological, soil, and vegetation units) can be partially or totally indeterminate and blurred; many human concepts like “the Indian Ocean” are implicitly vague. The second problem is that spatial data modeling mainly rests on Euclidean space and Euclidean geometry and hence on an infinite-precision arithmetic. This conflicts with the reality of finite-precision number systems available in computers. For example, an intersection point of two lines has to be rounded to the nearest grid point where the grid corresponds to the resolution of the number system used. It is left to the implementor to close this gap between theory and practice. This leads inevitably not only to numerical but especially to topological errors and thus to wrong query results. Hence, it is recommendable to incorporate the aspect of finite representations explicitly into a spatial data model and to design *finite resolution spatial data types* that can be integrated as attribute types into databases. *Image data modeling*, as the basis of image databases and image processing, suffers from the problem that previous attempts to design a consistent topological model for digital images have led to topological anomalies or have implied new unfavorable properties.

The goals of this paper are three-fold: first, we provide a conceptual model of crisp and fuzzy spatial objects (in this paper we will only describe regions) and investigate their topological properties. Second, we base our spatial objects on a finite resolution grid in order to take into account the discrete representations available in computers. These representations are especially interesting for fuzzy spatial objects since each contained, representable point has to be associated with a membership value. Third, this paper shall also be a contribution to achieve a uniform treatment of vector and raster data.

Section 2 compares with related work. Section 3 gives a unified view of vector and raster data, and Section 4 introduces a formalization of crisp and fuzzy regions based on a finite resolution grid. Section 5 gives a definition of the fuzzy geometric operations union, intersection, and difference, and Section 6 deals with topological predicates for discrete regions. Section 7 concludes the paper.

2 RELATED WORK

Mainly two kinds of *spatial vagueness* can be identified: *uncertainty* is traditionally equated with randomness and chance occurrence and relates either to a lack of knowledge about the position and shape of an object with an existing, real boundary or to the inability of measuring such an object precisely. *Fuzziness* is an intrinsic feature of an object itself and describes the vagueness of an object which certainly has an extent but which inherently cannot or does not have a precisely definable boundary (e.g., between a mountain and a valley). At least four alternatives are proposed as general design methods: (1) *exact models* (for example, (Clementini *et al.*, 1996; Erwig *et al.*, 1997b; Schneider, 1996)) which transfer type systems and concepts for spatial objects with sharp boundaries to objects with unclear boundaries, (2) *models based on rough sets* (Worboys, 1998) which work with lower and upper approximations of spatial objects, (3) *probabilistic models* (for example, (Burrough, 1996; Shibasaki, World Scientific, 1993)) which predominantly model positional and measurement uncertainty, and (4) *models based on fuzzy sets* (for example, (Altman, 1994; Burrough, 1996; Dutta, 1989; Schneider, 1999) which predominantly model fuzziness. The vagueness represented by fuzziness, in which we are only interested in this paper, does not describe the uncertainty of expectation like in probabilistic models but the vagueness resulting from the imprecision of the meaning of a concept. Examples of fuzzy spatial objects include mountains, valleys, biotopes, oceans. The only proposal of a discrete fuzzy data type relates to fuzzy regions (Altman, 1994) defined as a fuzzy set over $\mathbb{N}^2$. Each coordinate $(x,y) \in \mathbb{N}^2$ is associated with a value between 0 and 1 and describes the concentration of some feature attribute at that point. Unfortunately, the simple set property is insufficient since topological anomalies can arise, as we will see later.

In image data modeling and image processing in principle there has never been a problem to handle, that is, to visualize, values being of the same kind but having different intensity levels. Color shading and gray values are examples of such visualization methods. Thus, fuzziness has played a certain role. This can be seen from *fuzzy digital topology* (Rosenfeld, 1979), the fuzzy version of *digital topology* (Kong *et al.*, 1989), which has been applied to pixel structures.

Finite resolution spatial data, that is, spatial data defined over a discrete underlying geometric domain (for example, a homogeneous grid), has so far had a completely different significance for spatial and image information. It lies in the nature of image data modeling to handle discrete spatial data which is available as digital raster images. Digital topology (fuzzy digital topology) has been applied to describe the structure and the topological features of binary (finite-valued) images. In Section 3 we will show that these theories suffer from topological anomalies which make them inappropriate for our purposes. In spatial data modeling hardly any work has so far been based on finite resolution data. *Spatial data types* (see (Schneider, 1997) for an overview) proposed so far have all been based on exact two-dimensional Euclidean space. An exception is the *realm* concept (Güting *et al.*, 1993; Schneider, 1997). A realm replaces the Euclidean space with a discrete geometric basis and is intended to represent the entire underlying geometry of an application. It consists of a finite set of points and line segments which are defined over a discrete grid and which form a spatially embedded planar graph. On top of realms a comprehensive and coherent spatial type system called the *ROSE* *algebra* (Güting *et al.*, 1995; Schneider, 1997) and a concept of *vague regions* (Erwig *et al.*, 1997b) have been built. Each spatial object is described by a finite boundary representation and consists of a finite set of representable grid points.

The expositions so far indicate that in the end both spatial and image data modeling have (also conceptually) to deal with finite representations in order to guarantee implementability of spatial type systems in general and numerical robustness and topological consistency in particular. This understanding is a prerequisite for a uniform spatial information theory.

3 UNIFYING THE VIEW OF VECTOR AND RASTER DATA

In this section we deal with the transformation of vector data into finite resolution data (Section 3.1), with the benefits of point set topology and the deficiencies of digital topology (Section 3.2), and with grid partitions as the underlying discrete geometric domain for finite resolution spatial objects (Section 3.3).

3.1 Transforming Vector Data to Finite Resolution Data

The transformation of vector data to finite resolution data can be imagined as a three-step process. In a first step, we determine as the underlying discrete geometric domain a homogeneous *grid* given as a finite subset $\Omega = \{-n, \ldots, n\}^2 \subset \mathbb{Z}^2$ with an arbitrary but fixed and representable $n \in \mathbb{N}$. An element of Ω is called a *grid point*. The second step is to define spatial objects by boundary representations over this finite resolution grid so that their vertices are grid points. In a third step, a spatial object is represented as the finite collection of points (possibly with particular properties) enclosed by the object's boundary representation. This mapping is usually not invertible, that is, given only the finite point set of a spatial object, we are not able to uniquely derive its original boundary. As an example, Figure 1 shows three different boundary representations of regions enclosing the same point set. If we assume a very high resolution of Ω, that is, a very large n, the perturbations resulting from this step become neglectible. We can then accept these slight perturbations and consider all spatial objects with the same finite point set as being equal.

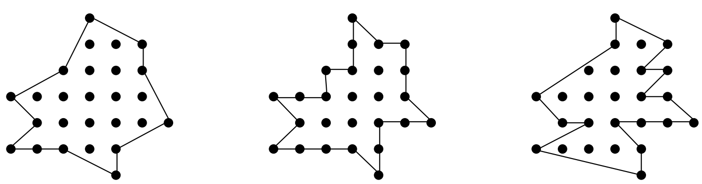


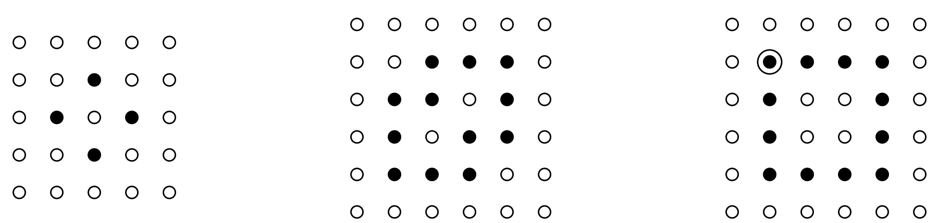
Figure 1: An example of three different regions with the same finite point set.

3.2 Point Set Topology and the Deficiencies of Digital Topology

In the Euclidean space, *point set topology* (Gaal, 1964) has turned out to be an appropriate theory to characterize topological relationships between spatial objects. Therefore, we would like to transfer its well-known and desirable properties to an appropriate *finite topology*. The starting point of point set topology is the notion of a topological space. Let X be a set and $\mathcal{T} \subseteq 2^X$ be a subset of the power set of X. The pair $(X, \mathcal{T})$ is called a *topological space* if the axioms (i) $X \in \mathcal{T}, \varnothing \in \mathcal{T}$, (ii) $U \in \mathcal{T}, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$, and (iii) $U \subseteq \mathcal{T} \Rightarrow \bigcup_{A \in U} A \in \mathcal{T}$ are satisfied. $\mathcal{T}$ is called a *topology* for X. The elements of $\mathcal{T}$ are called *open sets*, their complements in X *closed sets*. Point set topology mainly considers infinite point sets having the property that in an arbitrarily small neighborhood of a point infinitely many other such points exist. This contradicts the nature of a discrete grid-based point whose neighborhood contains at most a finite number of other points. Additionally, point set topology distinguishes different parts of a point set, namely its *boundary* ∂A, its *interior* A°, and its *exterior* A^-, which are pairwise disjoint. The union of A° and ∂A corresponds to the *closure* $\overline{A}$ of A. In $\mathbb{R}^2$ the Jordan Curve Theorem (Gaal, 1964) is valid which states that a simple closed and continuous curve divides the Euclidean plane into two connected components, the interior and the exterior. If a point is removed from the curve, then the remainder of the plane becomes connected.

Digital topology (Kong *et al.*, 1989) is the study of the topological properties of discrete images and is therefore a possible candidate for modeling topological properties of crisp and fuzzy discrete spatial objects. But the following short summary reveals its fundamental deficiences on a discrete domain compared to point set topology on a continuous domain. The underlying space of digital topology is the *digital plane* $\mathbb{Z}^2$. Let $S \subset \mathbb{Z}^2$. The points in S are then called *black* points, and the points in $\mathbb{Z}^2 - S$ are termed *white* points. Two main kinds of neighborhoods can be distinguished between grid points. The 4-*neighbors* of a point are its four horizontal and vertical neighbors. The 8-*neighbors* of a point are its four horizontal and vertical neighbors together with its four diagonal neighbors. A 4-*path* (8-*path*) between any two points $p, q \in S$ is a sequence of points $p = p_1, \ldots, p_n = q$ with $p_i \in S$ such that p_i is a 4-neighbor (8-neighbor) of $p_{i-1}, 1 < i \leq n$. A path is called *simple* if each of its points is exactly once in the path. A set S is 4-*connected* (8-*connected*) if for every pair of points p, q of S there is a 4-path (8-path) in S from p to q. A 4-*component* (8-*component*) of a set S is a greatest 4-connected (8-connected) subset of S.

The *connectivity paradox* (Kong *et al.*, 1989) illustrates a possible “topological paradox” and shows that the Jordan Curve Theorem is neither valid for 4-neighborhood nor for 8-neighborhood. If in Figure 2a 8-neighborhood is used for all pairs of points, then the black points form a discrete analog of a Jordan curve (simple closed path) but they do not separate the white points. The situation is not better for 4-neighborhood. The black points in Figure 2b determine a 4-connected simple closed path but there exist three 4-connected components for the remaining white points. Thus a digital version of the Jordan Curve Theorem holds in neither case. Moreover, if in Figure 2a 4-neighborhood is used for all pairs of points, then the black points are completely disconnected but still separate the set of white points into two components. If 4-neighborhood is used for all pairs of points and we have a 4-connected simple closed path like in Figure 2c, then the white points remain separated if we remove a corner (circled point) from the path.



(a) (b) (c)

Figure 2: Examples of connectivity paradoxes.

The problem of digital topology is that it does not refer to the definition of a topological space. Hence, it is impossible to transfer important topological notions like open set, open neighborhood, continuity, and many others, to a two-dimensional grid-based domain. Another drawback is that boundaries are two-dimensional structures, a fact that does not correspond to our expectation of spatial reality.

3.3 The Unification Step: Grid Partitions

The deficiencies of digital topology can be remedied if we return to classical topology. Then a solution comprises two main aspects: first, we form *unit areas* which relate to the regions between four adjacent grid points and which contain all and only their interior and boundary points. This measure ensures the existence of an underlying topological space and the avoidance of geometric anomalies. The consequence is that the intersection of two unit areas is either empty (if they are disjoint), or a zero-dimensional point (if they meet at a corner), or a one-dimensional unit segment (if they share a unit segment), or a two-dimensional unit area (if they are identical). Hence, second, we have to consider the digital plane as a structure consisting of elements of different dimensions. Such a structure is well-known in *algebraic topology* as a *cellular complex* (Croom, 1978) and has already been used in spatial data modeling (for example, (Egenhofer *et al.*, 1989; Frank *et al.*, 1986)) for decomposing space into a collection of irregular geometric shapes.

We briefly discuss some of the most important definitions. A *homeomorphism* (Croom, 1978) between two spatial objects is an invertible function from one object to another such that both the function and its inverse are continuous. This essentially means that there is a mapping of the points in the first object to the points in the second one (and vice versa) that preserves the concept of proximity. The *closed n-disc* with center x and radius $\varepsilon \in \mathbb{R}^+$ is the set of points in the n-dimensional Euclidean space with a distance from x less than or equal to ε. A closed *n-cell* is any spatial object homeomorphic (that is, topologically equivalent) to the closed n-disc. The surface of the closed $(n+1)$-disc with radius ε describes all points at a distance of exactly ε and is called the *n-sphere*. The *boundary* of an n-cell is that part of the n-cell mapped onto by the $(n-1)$-sphere by any homeomorphism. The *interior* of an n-cell is that part of its $n-1$-cells that do not belong to the boundary. An *open n*-cell is a closed n-cell without boundary. Examples of 0-cells are points, examples of 1-cells are continuous curves and straight segments, and examples of 2-cells are circles, triangles, and polygons. The boundary of a triangle, for example, consists of its three bounding straight segments. The boundary of each segment contains its two end points. All k-cells of an n-cell for $0 \leq k < n$ are called *faces* of the n-cell. A k-cell is said to *bound* an n-cell with $0 \leq k < n$ if the k-cell is a face of the n-cell.

An *n-dimensional cellular complex* (*n-complex, n-cell complex*) is a collection C of n-cells such that (i) C contains all faces of all elements of C (called *completeness of inclusion* in (Frank *et al.*, 1986)), and (ii) the intersection of two cells in C is either empty or a face of both cells (called *completeness of incidence* in (Frank *et al.*, 1986)). These two conditions correspond to the definition of a (finite) topological space so that a cellular complex is a finite topological space. From a data type point of view, this means that cellular complexes are closed under the set operations *union*, *intersection*, *difference*, and *complement*[1]. An n-complex of special interest is the punctured cell consisting of an n-cell from which a smaller n-cell has been cut out.

Actually, we return to the vector model of spatial databases where we can distinguish one-, zero-, and two-dimensional spatial elements, too. In our case we use cells and cell complexes to obtain a *regular* cellular decomposition of the plane which we call a *grid partition*. The grid partition restricts the set of possible structures in $\mathbb{R}^2$ to collections of regularly shaped *grid units*. This measure transfers the topological properties of $\mathbb{R}^2$ to the grid partition since the grid partition is embedded into the continuous Euclidean plane being a topological space. Hence, the grid partition turns out to be able to replace the image (or raster) model of $\mathbb{Z}^2$.

We now define the notion of a grid unit as the central component of a grid partition. Let $q = (x,y) \in \mathbb{R}^2$, and let ν be a *normalization* function with $\nu((x,y)) = (x+\frac{1}{2}, y+\frac{1}{2})$. We extend ν to subsets $A \subseteq \mathbb{R}^2$ by $\nu(A) = \{\nu(q) \mid q \in A\}$. Moreover, for a point $p = (i,j) \in \Omega$ with $i,j \notin \{-n,n\}$ we define $p_0(i,j) = (i-\frac{1}{2}, j-\frac{1}{2}), p_1(i,j) = (i+\frac{1}{2}, j-\frac{1}{2}), p_2(i,j) = (i+\frac{1}{2}, j+\frac{1}{2})$, and $p_3(i,j) = (i-\frac{1}{2}, j+\frac{1}{2})$. A *grid unit* for a point $p = (i,j) \in \Omega$ is the triple $G(p) = (C(p), E(p), V(p))$ such that

(i) $C(p) = \{\nu(]i-\frac{1}{2}, i+\frac{1}{2}[\times]j-\frac{1}{2}, j+\frac{1}{2}[)\}$

(ii) $E(p) = \{\nu(]p_t(i,j), p_{(t+1) \bmod 4}(i,j)[) \mid 0 \leq t \leq 3\}$

[1] An interesting observation is that a realm (Güting *et al.*, 1993; Schneider, 1997) also forms a cellular complex and that it hence is a finite topological space.

(iii) $V(p) = \{\nu(p_t(i,j)) \mid 0 \le t \le 3\}$

Since the points $p_t(i,j)$ are not elements of Ω and thus not representable in our discrete domain, the function ν performs a homeomorphic translation operation to representable coordinates in Ω. Hence, each point (i,j) is mapped to the grid unit (unit square) $G((i,j))$ with the left lower bound (i,j) and the right upper bound $(i+1, j+1)$. A grid unit consists of three parts: $C(p)$ contains its axis-parallel, open quadrangle of unit length 1 (called *unit area*) as a 2-cell and is modeled as a singelton set. To denote the point set of this 2-cell we need a further notation. Let A, B be sets and $f : A \to \mathcal{P}(B)$ be a function. If we are sure that, for example, $f(a)$ yields a singleton set, we write $f^!(a)$ to denote this single element, that is, $f(a) = \{b\} \Rightarrow f^!(a) = b$. Now we can express the point set of $C(p)$ as *points*$(C(p)) := C^!(p)$. $E(p)$ contains its four axis-parallel, open unit segments (called *edges*) as

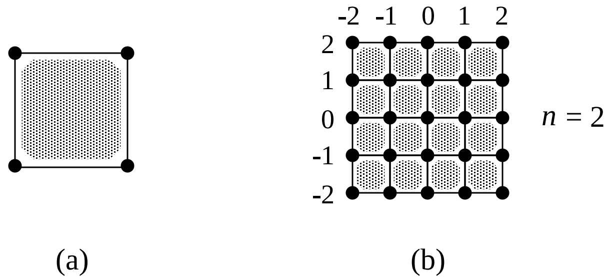


Figure 3: A grid unit as a 2-cell (a) and a grid partition as a 2-cell complex (b).

1-cells, and its point set is *points*$(E(p)) := \bigcup_{S \in E(p)} S$. $V(p)$ contains its four corners (called *points* or *vertices*) as 0-cells, and its point set is *points*$(V(p)) := V(p)$. We have deliberately defined that every k-cell is open and does hence not contain its boundary, that is, *points*$(C(p)) \cap$ *points*$(E(p)) \cap$ *points*$(V(p)) = \emptyset$. Thus, each point of $G(p)$ belongs either to a 0-cell, a 1-cell, or a 2-cell. The point set *points*$(C(p)) \cup$ *points*$(E(p)) \cup$ *points*$(V(p))$ describes the complete point set of the grid unit. Figure 3a shows the structure of a grid unit as a cellular complex. The *grid partition*[2] over Ω is the set $G(\Omega) = \{G(p) \mid p \in \Omega\}$. Moreover, we define $C(\Omega) = \{C(p) \mid p \in \Omega\}$, $E(\Omega) = \{E(p) \mid p \in \Omega\}$, and $V(\Omega) = \{V(p) \mid p \in \Omega\}$. An analogous definition holds for $G(\mathbb{Z}^2)$. Figure 3b shows an example of a grid partition as a cellular complex.

In summary one can say that in a two-stage process we have substituted the Euclidean plane $\mathbb{R}^2$ for the digital plane $\mathbb{Z}^2$ and the latter due to topological weaknesses for the grid partition $G(\mathbb{Z}^2)$. The result is a regular cellular decomposition of $\mathbb{R}^2$ (or a regular realm in the sense of (Güting *et al.*, 1995; Schneider, 1997)), which is also discrete like $\mathbb{Z}^2$, but moreover preserves the topological features of $\mathbb{R}^2$. In image modeling the 2-cells of $G(\mathbb{Z}^2)$ correspond to pixels. We have here the problem that 0-cells and 1-cells are not realized in hardware devices like monitor screens and image memories, but grid partitions can at least serve as a conceptual model for topologically consistent pictures in image processing. In spatial (that is, vector) modeling, the conceptual extensions are realizable without difficulties.

4 FINITE RESOLUTION CRISP AND FUZZY REGIONS

In this section we in a very general way define a spatial data type for crisp and fuzzy region objects as parts of the discrete geometric domain $G(\Omega)$. Generality here especially implies that the data type

[2]The notion of grid partition differs from the notion of *spatial partition* (map, coverage) given, for example, in (Erwig *et al.*, 1997a). In a spatial partition, boundaries of adjacent areas with the same label are eliminated, and the areas are merged together.

is closed under the geometric operations *union*, *intersection*, and *difference* (as well as under other spatial operations). This means, in particular, that a region may have holes and that it may consist of several components.

Section 4.1 introduces some concepts of fuzzy set theory as far as they are relevant in this context. Section 4.2 introduces crisp and fuzzy regions. It turns out that the crisp region type is a special instance of the fuzzy region type.

4.1 Fuzzy Sets and Discrete Membership Functions

Fuzzy set theory (Zadeh, 1965) is an extension and generalization of Boolean set theory. Let X be a classical (crisp) set of objects, called the *universe* (*of discourse*). Membership in a classical subset A of X can then be described by the *characteristic function* $\chi_A : X \to \{0,1\}$ such that for all $x \in X$ holds $\chi_A(x) = 1$ if and only if $x \in A$ and $\chi_A(x) = 0$ otherwise. This function can be generalized such that all elements of X are mapped to the real interval [0,1] indicating the *degree of membership* of these elements in the set in question. We call $\mu_{\tilde{A}} : X \to [0,1]$ the *membership function* of $\tilde{A}$, and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \,|\, x \in X\}$ is called a *fuzzy set* in X. Elements $x \in X$ that do (not) belong to $\tilde{A}$ get the membership value $\mu_{\tilde{A}}(x) = 1$ (0). We also allow the notations $\tilde{x} := (x, \mu_{\tilde{A}}(x)) \in \tilde{A}$ and $\mu(\tilde{x}) := \mu_{\tilde{A}}(x)$. Regarding the set operations and the set inclusion for fuzzy sets we follow the definitions in (Zadeh, 1965). Let $\tilde{A}$ and $\tilde{B}$ be fuzzy sets in X. Then

(i) $\neg\tilde{A} = \{(x, \mu_{\neg\tilde{A}}(x)) \mid x \in X, \mu_{\neg\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)\}$

(ii) $\tilde{A} \subseteq \tilde{B} \Leftrightarrow \forall\, x \in X : \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$

(iii) $\tilde{A} \cap \tilde{B} = \{(x, \mu_{\tilde{A}\cap\tilde{B}}(x)) \mid x \in X \,\wedge\, \mu_{\tilde{A}\cap\tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))\}$

(iv) $\tilde{A} \cup \tilde{B} = \{(x, \mu_{\tilde{A}\cup\tilde{B}}(x)) \mid x \in X \,\wedge\, \mu_{\tilde{A}\cup\tilde{B}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))\}$

(v) $\tilde{A} - \tilde{B} = \tilde{A} \cap \neg\tilde{B}$

A [*strict*] α-*cut* or [*strict*] α-*level set* of a fuzzy set $\tilde{A}$ for a specified value α is the crisp set A_α $[A^*_\alpha] = \{x \in X \mid \mu_{\tilde{A}}(x) \geq [>]\, \alpha \,\wedge\, 0 \leq \alpha \leq [<]\, 1\}$. The strict α-cut for $\alpha = 0$ is called *support* of $\tilde{A}$, i.e., $supp(\tilde{A}) = A^*_0$. For a fuzzy set $\tilde{A}$ and $\alpha, \beta \in [0,1]$ we obtain $X = A_0$ and $\alpha < \beta \Rightarrow A_\alpha \supseteq A_\beta$. The set of all levels $\alpha \in [0,1]$ that represent distinct α-cuts of a given fuzzy set $\tilde{A}$ is called the *level set* $\Lambda_{\tilde{A}}$ of $\tilde{A}$: $\Lambda_{\tilde{A}} = \{\alpha \in [0,1] \mid \exists\, x \in X : \mu_{\tilde{A}}(x) = \alpha\}$. We also introduce a special notation for subsets of the power set $\mathcal{P}(\Lambda)$ containing a particular element $\alpha_k \in \Lambda$ and being of constrained size: for $1 \leq k \leq m$ and $t \in \mathbb{N}$ we define $\Lambda^{k,t} := \{A \in \mathcal{P}(\Lambda) \,|\, \alpha_k \in A, 1 \leq |A| \leq t\}$.

Since we are aiming at finite representations, the definition of a membership function is too general. The universe of discourse we are interested in is the finite grid partition $X = G(\Omega)$. Since the interval $[0,1]$ represents an infinite set of membership values, we have to restrict it to a finite and thus representable set $\Lambda = \{\alpha_1, \ldots, \alpha_m\}$ for some $m \in \mathbb{N}$ with $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1} < \alpha_m = 1$. Accordingly, in our context we have *discrete membership functions*, and the discrete membership function for any fuzzy set $\tilde{A}$ is here defined as $\mu_{\tilde{A}} : (C(\Omega) \cup E(\Omega) \cup V(\Omega)) \to \mathcal{P}(\Lambda)$. This means that a grid unit is always associated with a *set of membership values*, which we here also call *labels*.

The selection of membership values is not arbitrary for the faces of fuzzy regions due to the topological interdependence of 0-cells, 1-cells, and 2-cells of the grid units of the grid partition. Another important aspect motivated by spatial phenomena like air pollution or temperature distributions is that we are interested in modeling "smooth" or "continuous" transitions within the interiors of regions. This feature resembles grey values in images which are used to visualize different levels of intensity of an attribute.

4.2 Formal Definition of Fuzzy Regions

The aim of this section is to develop and formalize our concept of *discrete crisp and fuzzy regions*. A detailed classification of fuzzy regions can be found in (Schneider, 1999). Our main interest concerns a "structured" view of discrete fuzzy regions. We start with the definition of a *fuzzy grid unit*. For this purpose we have to appropriately assign labels to its 2-cell, its four 1-cells, and its four 0-cells. The membership value of a grid unit as a whole is dominated by the 2-cell so that all of its 0- and 1-cells obtain this value.

A *fuzzy grid unit* for a point $p \in \Omega$ and a label $\{\alpha_k\}$ is a triple $G_f^k(p) = (C_f^k(p), E_f^k(p), V_f^k(p))$ such that $C_f^k(p) = C(p) \times \Lambda^{k,1}$, $E_f^k(p) = E(p) \times \Lambda^{k,2}$, and $V_f^k(p) = V(p) \times \Lambda^{k,4}$ for some $1 \leq k \leq m$. That is, the 2-cell of a fuzzy grid unit is marked with a singelton label set, each 1-cell with a set of up to two labels, and each 0-cell with a set of up to four labels. Note that we also permit $\alpha_1 = 0$ as part of a label; it indicates the non-existence (of a component) of a fuzzy grid unit. Due to m possible α-labels, for each point $p \in \Omega$ we can define m different fuzzy grid units (including a non-existing unit) which we collect in $G_f(p) = \bigcup_{k=1}^{m}\{G_f^k(p)\}$. We can also gather all possible 2-cells of a fuzzy grid unit for a point p in $C_f(p) = \bigcup_{k=1}^{m} C_f^k(p)$, all possible 1-cells in $E_f(p) = \bigcup_{k=1}^{m} E_f^k(p)$, and all possible 0-cells in $V_f(p) = \bigcup_{k=1}^{m} V_f^k(p)$. Finally, the set of all fuzzy grid units over Ω is $G_f(\Omega) = \bigcup_{p\in\Omega} G_f(p)$, the set of all *fuzzy unit areas* is $C_f(\Omega) = \bigcup_{p\in\Omega} C_f(p)$, the set of all *fuzzy edges* is $E_f(\Omega) = \bigcup_{p\in\Omega} E_f(p)$, and the set of all *fuzzy vertices* is $V_f(\Omega) = \bigcup_{p\in\Omega} V_f(p)$.

Let $u_1 = G_f^k(p), u_2 = G_f^k(q) \in G_f(\Omega)$ for some $1 < k, l \leq m$. We define the following topological predicates on fuzzy grid units:

(i)	u_1 and u_2 0-*meet*	$:\Leftrightarrow$	$C(p) \cap C(q) = \varnothing \;\wedge\; E(p) \cap E(q) = \varnothing$ $\wedge\; V(p) \cap V(q) \neq \varnothing$
(ii)	u_1 and u_2 1-*meet*	$:\Leftrightarrow$	$C(p) \cap C(q) = \varnothing \;\wedge\; E(p) \cap E(q) \neq \varnothing$ $\wedge\; V(p) \cap V(q) \neq \varnothing$
(iii)	u_1 and u_2 are *area-disjoint*	$:\Leftrightarrow$	$C(p) \cap C(q) = \varnothing$
(iv)	u_1 and u_2 are *edge-disjoint*	$:\Leftrightarrow$	$C(p) \cap C(q) = \varnothing \;\wedge\; E(p) \cap E(q) = \varnothing$

As the definitions show, these predicates are independent of the labels of a fuzzy grid unit. The predicate 0-*meet* expresses that both fuzzy grid units have a zero-dimensional vertex in common while the predicate 1-*meet* describes that both fuzzy grid units meet in a one-dimensional edge and its two bounding vertices. The predicate *area-disjoint* allows both units to share a common edge together with the two bounding vertices whereas the predicate *edge-disjoint* only allows them to share a common vertex.

A fuzzy grid unit complex (an example is given in Figure 4a) is a connected set of fuzzy grid units $c = \{u_1, \ldots, u_n\} \subseteq G_f(\Omega)$ with $u_i = G_f^{k_i}(p_i), p_i \in \Omega$, $1 < k_i \leq m$ such that

(i) $\forall 1 \leq i < j \leq n : u_i$ and u_j are *area-disjoint*

(ii) $\forall 1 \leq i \leq n \; \exists 1 \leq j \leq n, i \neq j : u_i$ and u_j 1-*meet*

(iii) The point set $\partial \bigcup_{i=1}^{n}(\mathit{points}(C(p_i)) \cup \mathit{points}(E(p_i)) \cup \mathit{points}(V(p_i)))$ is a simple polygon.

(iv) $\forall 1 \leq i < j \leq n : (u_i$ and u_j 1-*meet* $\wedge\; E(p_i) \cap E(p_j) = \{e\} \;\wedge\; V(p_i) \cap V(p_j) = \{q_1, q_2\}) \Rightarrow (\mu(\tilde{e}) := \mu(C(p_i)) \cup \mu(C(p_j)), \mu(\tilde{q}_1) := \mu(\tilde{q}_1) \cup \mu(C(p_i)) \cup \mu(C(p_j)), \mu(\tilde{q}_2) := \mu(\tilde{q}_2) \cup \mu(C(p_i)) \cup \mu(C(p_j)))$

(v) $\forall 1 \leq i < j \leq n : (u_i$ and u_j 0-*meet* $\wedge\; V(p_i) \cap V(p_j) = \{q\}) \Rightarrow \mu(\tilde{q}) := \mu(\tilde{q}) \cup \mu(C(p_i)) \cup \mu(C(p_j)))$

(vi) Let $\overline{c} = \{v_1, \ldots, v_t\} = G_f(\Omega) \setminus c$ with $v_j = G^l_f(q_j), q_j \in \Omega, 1 \leq l \leq m$. $\forall\, 1 \leq i \leq n\, \forall\, 1 \leq j \leq t$: (u_i and v_j 1-*meet* $\wedge\, E(p_i) \cap E(q_j) = \{e\} \wedge V(p_i) \cap V(q_j) = \{s_1, s_2\}) \Rightarrow (\mu(\tilde{e}) := \mu(C(p_i)) \cup \{\alpha_1\}, \mu(\tilde{s}_1) := \mu(\tilde{s}_1) \cup \mu(C(p_i)) \cup \{\alpha_1\}, \mu(\tilde{s}_2) := \mu(\tilde{s}_2) \cup \mu(C(p_i)) \cup \{\alpha_1\})$

(vii) Let $\overline{c} = \{v_1, \ldots, v_t\} = G_f(\Omega) \setminus c$ with $v_j = G^l_f(q_j), q_j \in \Omega, 1 \leq l \leq m$. $\forall\, 1 \leq i \leq n\, \forall\, 1 \leq j \leq t$: (u_i and v_j 0-*meet* $\wedge\, V(p_i) \cap V(q_j) = \{s\}) \Rightarrow \mu(\tilde{s}) := \mu(\tilde{s}) \cup \mu(C(p_i)) \cup \{\alpha_1\})$

Conditions (i) and (ii) formulate the partition character and the connectivity property, respectively. Condition (iii) prohibits holes in the complex. Condition (iv) applies the *label union rule* to all common edges and vertices of neighbored grid units. Each edge has one or two labels. It has two labels if the labels of the units are different; otherwise it has one label. The bounding vertices of the edge obtain the labels of the edge additionally to the labels they already have from other bounded edges; they can have at most four labels. Condition (v) appropriately labels the vertex of two grid units that 0-meet. Conditions (vi) and (vii) deal with all vertices and edges that bound grid units, that belong also to the complement of the complex, and that especially carry the label $\{\alpha_1\} = \{0\}$ indicating their non-belonging to the complex. Edges of this kind have exactly two labels; vertices of this kind can have up to four labels. We call conditions (iv) to (vii) *label correction*.

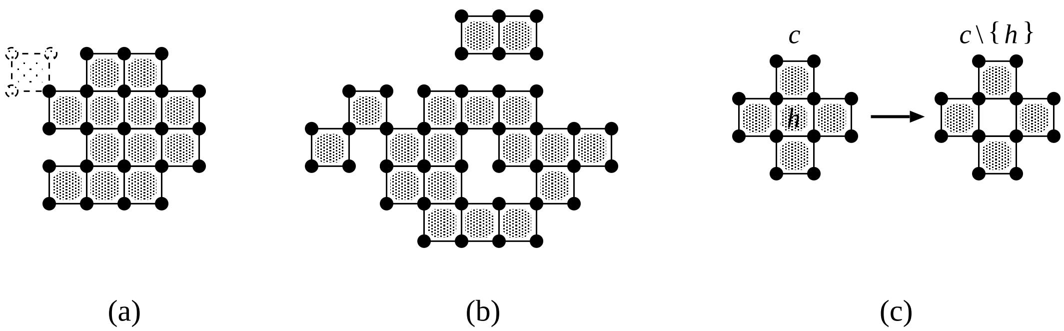


Figure 4: An example of a fuzzy grid complex is shown in (a). Adding the dotted grid unit would invalidate the complex. Four faces are shown in (b). The example in (c) shows a simple situation where c and h are complexes but $c \setminus \{h\}$ is not a fuzzy face.

Let $CO_f(\Omega) = \{c \subseteq G_f(\Omega) \,|\, c \text{ is a fuzzy grid unit complex}\}$, and let $c_1 = \{u_1, \ldots, u_n\}$, $c_2 = \{v_1, \ldots, v_t\} \in CO_f(\Omega)$ with $u_i = G^{k_i}_f(p_i), p_i \in \Omega$, and $v_j = G^{l_j}_f(q_j), q_j \in \Omega, 1 < k_i, l_j \leq m$. We define the following two predicates:

(i) c_1 and c_2 are *edge-disjoint* $:\Leftrightarrow$ $\forall\, 1 \leq i \leq n\, \forall\, 1 \leq j \leq t$:
u_i and v_j are *edge-disjoint*

(ii) c_1 *edge-inside* c_2 $:\Leftrightarrow$ $\forall\, 1 \leq i \leq n$:
$(\exists\, 1 \leq j \leq t : p_i = q_j \wedge \mu^!_{u_i}(C(p_i)) \leq \mu^!_{v_j}(C(q_j))) \wedge$
$(\forall\, \tilde{e} \in E^{k_i}_f(p_i) : \alpha_1 = 0 \notin \mu(\tilde{e}))$

The predicate *edge-disjoint* implies that two fuzzy grid unit complexes may only share single vertices but no edges. Otherwise, we could merge them together into a single fuzzy grid unit complex. The predicate *edge-inside* first checks whether each grid unit of c_1 is contained in c_2 and whether the label of the 2-cell of c_1 is not greater than the label of the 2-cell of c_2. A second condition additionally requires that the edges of all fuzzy grid units of c_1 are not labeled with 0. This, in particular, means that the vertices of the grid units of c_1 are allowed to have the label 0.

Moreover, this condition identifies the corresponding grid units of c_1 in c_2 and tests whether c_2's edges contain the label α_1. If this is not the case, then we know that c_1 lies properly in c_2, otherwise c_1 contains at least one grid unit lying on the “thick boundary” of a grid unit in c_2.

A *fuzzy face* f (see Figure 4b) is a pair (c,H) with $c \in CO_f(\Omega), H = \{h_1, \ldots, h_n\}$ with $h_i \in CO_f(\Omega)$ such that the following conditions hold (let $U(H) = \bigcup_{1 \leq i \leq n} h_i$, let $p,q \in \Omega, 1 < k,l \leq m$, and let $U_f(f)$ denote the set of all grid units of f):

(i) $\forall u \in U_f(H) : u = G_f^k(p) \Rightarrow \mu_u^!(C(p)) = \alpha_m = 1$ (or: $k = m$)

(ii) $\forall u \in U_f(H)\ \exists v \in c : (u = G_f^k(p) \wedge v = G_f^l(q)) \Rightarrow (p = q \wedge \mu_v^!(C(q)) = \mu_u^!(C(p))$

(iii) $\forall 1 \leq i \leq n : h_i$ *edge-inside* c

(iv) $\forall 1 \leq i < j \leq n : h_i$ and h_j are *edge-disjoint*

(v) Each fuzzy grid unit complex is either equal to c or to one of the elements in H (that is, no other fuzzy grid unit complex can be formed from the units of f)

(vi) $U_f(f) = c \setminus U_f(H)$

(vii) $\forall u \in U_f(f)\ \exists v \in U_f(f), u \neq v : u$ and v 1-*meet*

Conditions (i) and (ii) show the problem of defining a concept of "fuzzy holes" (see also the discussion in (Schneider, 1999)). In fact, there are only crisp holes, since only they can express parts enclosed by a fuzzy grid unit complex and do definitely not lie within the interior of c. We therefore have to require (conceptually) that the units composing the holes are crisp in c, too. Otherwise, condition (iii) would not work which tests whether all holes are edge-inside c. A hole is only allowed to share vertices and not edges with the exterior of c; otherwise we would form a "bay" in c, and we should have omitted the hole unit from the structure definition of c before. Condition (iv) requires that any two holes share at most vertices and not edges; otherwise we could merge them together into one hole. Condition (v) ensures uniqueness of representation, that is, there are no two different interpretations of a set of fuzzy grid units as sets of fuzzy faces. Condition (vi) describes the set of units belonging to f. Unfortunately, the set difference can yield a collection of fuzzy grid units which is not connected (see Figure 4c). The connectivity property is required in condition (vii).

Let $F_f(\Omega)$ denote the set of all fuzzy faces over Ω, and let $f = (f_0, \overline{F}), g = (g_0, \overline{G}) \in F_f(\Omega)$. We then define the predicate:

$$\begin{aligned} f \text{ and } g \text{ are } \textit{edge-disjoint} \;:\Leftrightarrow\; & f_0 \text{ and } g_0 \text{ are } \textit{edge-disjoint} \vee \\ & \exists \overline{g} \in \overline{G} : f_0 \textit{ edge-inside } \overline{g} \vee \exists \overline{f} \in \overline{F} : g_0 \textit{ edge-inside } \overline{f} \end{aligned}$$

The "structured view" of a fuzzy spatial data type for *discrete fuzzy regions* called *fregion* is based on fuzzy faces and is defined as follows:

$$\textit{fregion} = \{F \subseteq F_f(\Omega) \mid \forall f, g \in F : f \text{ and } g \text{ are } \textit{edge-disjoint}\}$$

The "flat view" emphasizes fuzzy grid units as the basic component of fuzzy regions:

$$\textit{fregion} = \{U \subseteq G_f(\Omega) \mid \exists F \subseteq F_f(\Omega) : U = \bigcup_{f \in F} U_f(f)\}$$

Given $F \in$ *fregion*, let $U_f(F) = \bigcup_{f \in F} U_f(f)$ be the set of all fuzzy grid units of F. We can then simply define the type *region* for discrete crisp regions as:

$$\textit{region} = \{F \in \textit{fregion} \mid \forall u = G_f^k(p) \in U_f(F), p \in V(\Omega), 1 < k \leq m : \mu_u^!(C(p)) = 1\}$$

Last but not least, we can give a definition for a special subtype of discrete fuzzy regions. This type called *sfregion* (for *smooth fuzzy regions*) takes into account "smooth" (that is, gradual) transitions within the interior of a region and models a kind of stepwise continuation ("discrete continuity"):

$$\begin{aligned} \textit{sfregion} = \{F \in \textit{fregion} \mid\ & \forall u = G_f^k(p), v = G_f^l(q) \in U_f(F), p, q \in V(\Omega), 1 < k, l \leq m : \\ & u \text{ and } v \text{ 1-}\textit{meet} \Rightarrow |k - l| \leq 1\} \end{aligned}$$

This means that two adjacent grid units have either the same label or two neighbored labels in the ordered sequence of labels of Λ. This strategy models stepwise continuation. Figure 5 shows a schematic example.

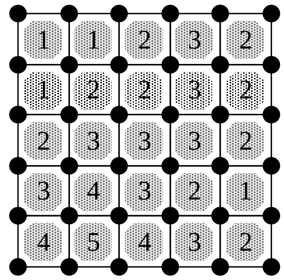


Figure 5: An example of the labeling strategy in a smooth fuzzy region. The label i stands for α_i.

Note that the grid units of a fuzzy region that carry the same label or a subcollection of labels form a subregion of the fuzzy region.

Obviously, *region* $\subset$ *sfregion* $\subset$ *fregion* holds.

5 GEOMETRIC UNION, INTERSECTION, AND DIFFERENCE

We will now define the three geometric operations $union_f$, $intersection_f$, and $difference_f$ which all have the signature $fregion \times fregion \to fregion$ and whose definition is based on the flat view. In case of $union_f$ we gather all grid units contained in the two fuzzy regions, label common 2-cells of the grid units with the maximum membership value, and adjust the label sets of bounding fuzzy vertices and edges according to the label correction mechanism. Let $F_1, F_2, F \in fregion$. The union operation is then defined on the basis of the union operation on fuzzy sets, and we obtain $F = union_f(F_1, F_2)$ such that

$$\begin{array}{rl}
\text{(i)} & \forall\, p \in \Omega\ \forall\, 1 < k,l \leq m : \\
& ((\exists s = G_f^k(p) \in U_f(F_1)\ \wedge\ \exists t = G_f^l(p) \in U_f(F_2)) \Rightarrow \exists u = G_f^{\max(k,l)}(p) \in U_f(F))\ \vee \\
& ((\exists s = G_f^k(p) \in U_f(F_1)\ \wedge\ \nexists t = G_f^l(p) \in U_f(F_2)) \Rightarrow \exists u = G_f^k(p) \in U_f(F))\ \vee \\
& ((\nexists s = G_f^k(p) \in U_f(F_1)\ \wedge\ \exists t = G_f^l(p) \in U_f(F_2)) \Rightarrow \exists u = G_f^l(p) \in U_f(F)) \\
\text{(ii)} & \text{Label correction applied to } U_f(F).
\end{array}$$

If F_1 and F_2 have a fuzzy grid unit u in common, u is associated in F with the maximum membership value of both units. Correspondingly, the operation $intersection_f(F_1, F_2)$ rests on the intersection of fuzzy sets und assigns the minimum membership value to fuzzy grid units that are part of both regions. Condition (i) from above has to be replaced with

$$\begin{array}{rl}
\text{(i)} & \forall\, p \in \Omega\ \forall\, 1 < k,l \leq m : \\
& ((\exists s = G_f^k(p) \in U_f(F_1)\ \wedge\ \exists t = G_f^l(p) \in U_f(F_2)) \Rightarrow \exists u = G_f^{\min(k,l)}(p) \in U_f(F))
\end{array}$$

The meaning of the operation $difference_f$ is not defined with the aid of the difference on fuzzy sets (that is, $P \setminus Q \neq P \cap \neg Q$), since the right side of the inequality does not make great sense in the spatial context. We replace condition (i) from above with

$$\begin{array}{rl}
\text{(i)} & \forall\, p \in \Omega\ \forall\, 1 < k,l \leq m : \\
& (\exists s = G_f^k(p) \in U_f(F_1)\ \wedge\ \exists t = G_f^l(p) \in U_f(F_2)\ \wedge\ k > l) \Rightarrow \\
& \quad (\exists u = G_f^w(p) \in U_f(F)\ \wedge\ w = \min\{v \in \{2, \ldots, m\} \mid \alpha_v \geq \alpha_k - \alpha_l\}
\end{array}$$

The idea is that the membership value of a unit area is diminished (if possible) by another unit area having the same location. In the crisp case this leads to a total elimination of the first unit area and in the fuzzy case to a partial elimination of the first unit area. $E(F)$ and $V(F)$ are defined as for $union_f$.

6 TOPOLOGICAL PREDICATES

In this section we deal with topological predicates for crisp and fuzzy spatial objects. Predicates computing topological relationships between spatial objects are very important for spatial databases, GIS, and image databases. In particular, they are needed in spatial query languages where they are employed as part of a filter condition in a query. For the Euclidean space, topological relationships have been studied very intensively. An important approach rests on the so-called 9-*intersection model* (Egenhofer *et al.*, 1989; Egenhofer, 1989) from which a canonical collection of topological relationships can be derived for each combination of spatial types. The model is based on the nine possible intersections of boundary (∂A), interior (A°), and exterior (A^-) of a spatial object with the corresponding components of another object. Each intersection is tested with regard to the topologically invariant criteria of emptiness and non-emptiness. This can be expressed for two spatial objects A and B by evaluating the following matrix:

$$\begin{pmatrix} \partial A \cap \partial B \neq \varnothing & \partial A \cap B^\circ \neq \varnothing & \partial A \cap B^- \neq \varnothing \\ A^\circ \cap \partial B \neq \varnothing & A^\circ \cap B^\circ \neq \varnothing & A^\circ \cap B^- \neq \varnothing \\ A^- \cap \partial B \neq \varnothing & A^- \cap B^\circ \neq \varnothing & A^- \cap B^- \neq \varnothing \end{pmatrix}$$

For this matrix $9^2 = 81$ different configurations are possible from which only a certain subset makes sense depending on the combination of spatial objects just considered. In this paper we will only deal with regions. A restriction of the 9-intersection model is that the regions considered must be homeomorphic to the closed disc, that is, they must be connected and are not allowed to have holes. We call this subtype of regions *region'*, and if we speak of regions in this section, we relate to them in the restricted sense just described. For two regions, eight meaningful configurations have been identified which lead to the eight predicates called *disjoint*, *meet*, *overlap*, *equal*, *inside*, *contains*, *covers*, and *coveredBy*. Each predicate is associated with a unique intersection matrix so that all predicates are mutually exclusive and complete with regard to the topologically invariant criteria of emptiness and non-emptiness:

$$\underset{\textit{disjoint}}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\textit{meet}}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\textit{overlap}}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\textit{equal}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\underset{\textit{inside}}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\textit{contains}}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\textit{covers}}{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\textit{coveredBy}}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}$$

Since our crisp and fuzzy spatial objects are part of the grid partition and since the grid partition is embedded into the Euclidean space and is thus a topological space, the spatial objects (and the grid partition) enjoy the topological properties of $\mathbb{R}^2$. Hence, it is permissible to apply the 9-intersection model to discrete crisp and fuzzy spatial objects. Unfortunately, we do not have, so far, concepts of boundary, interior, and exterior in our model. Consequently, we have to investigate in which manner fuzzy unit areas, edges, and vertices contribute to these three topological concepts.

Let $F \in$ *fregion*. We have already defined $U_f(F)$ as the set of all fuzzy grid units of F. Moreover, we define $C_f(F) = \{\tilde{c} \in C_f(\Omega) \,|\, \exists u = G_f^k(p) \in U_f(F), p \in \Omega, 1 < k \leq m : \tilde{c} = C_f^k(p)\}$ as the set of all fuzzy unit areas, $E_f(F) = \{\tilde{e} \in E_f(\Omega) \,|\, \exists u \in U_f(F) : \tilde{e}$ bounds $u\}$ as the set of all fuzzy edges and $V_f(F) = \{\tilde{p} \in V_f(\Omega) \,|\, \exists u \in U_f(F) : \tilde{p}$ bounds $u\}$ as the set of all fuzzy points of F. Restricting now F to exclusively *crisp* regions (that is, $F \in$ *region'*), we obtain crisp unit areas, edges, and vertices[3] carrying the label $\{1\}$. We specify how interior, boundary, and exterior of F can be expressed by them.

$$\begin{array}{lll}
\mathcal{C}(\partial F) = \varnothing & \mathcal{C}(F^\circ) = C(F) & \mathcal{C}(F^-) = C(\Omega) \setminus C(F) \\
\mathcal{E}(\partial F) = \{e \in E(F) \,|\, 0 \in \mu(e)\} & \mathcal{E}(F^\circ) = E(F) \setminus \mathcal{E}(\partial F) & \mathcal{E}(F^-) = E(\Omega) \setminus E(F) \\
\mathcal{V}(\partial F) = \{p \in V(F) \,|\, 0 \in \mu(p)\} & \mathcal{V}(F^\circ) = V(F) \setminus \mathcal{V}(\partial F) & \mathcal{V}(F^-) = V(\Omega) \setminus V(F)
\end{array}$$

This enables us now to denote the three topological sets of a crisp region F occurring in the intersection matrix as:

$$\begin{array}{lcl}
\partial F & = & \mathcal{C}(\partial F) \cup \mathcal{E}(\partial F) \cup \mathcal{V}(\partial F) \\
F^\circ & = & \mathcal{C}(F^\circ) \cup \mathcal{E}(F^\circ) \cup \mathcal{V}(F^\circ) \\
F^- & = & \mathcal{C}(F^-) \cup \mathcal{E}(F^-) \cup \mathcal{V}(F^-)
\end{array}$$

These sets can now be used to compute the intersection sets in the matrix. We must, of course, pay attention to the fact that only sets of compatible types can be combined. For example, the interior-interior intersection between two regions F and G can and must be limited to the following computation:

$$F^\circ \cap G^\circ = (\mathcal{C}(F^\circ) \cap \mathcal{C}(G^\circ)) \cup (\mathcal{E}(F^\circ) \cap \mathcal{E}(G^\circ)) \cup (\mathcal{V}(F^\circ) \cap \mathcal{V}(G^\circ))$$

This amounts to 27 intersection sets that apparently have to be calculated. But this number can be reduced to nine if we look at the possible dimensions of the entries in the intersection matrix. The observation (with one exception) is that the dimension of the intersection of two spatial components having dimension n and m, respectively, is equal to $\min(n,m)$, or the intersection is empty. The intersection of two open areas (interior or exterior) is two-dimensional or empty. This leads to four cases. The intersection of a boundary with an open area (interior or exterior) is one-dimensional or empty. This leads also to four cases. An exception is the boundary-boundary intersection which can have a one-dimensional (common edges) or a zero-dimensional (common vertices) result, or which can be empty. If this intersection has common edges, then this intersection has also common vertices, namely at least those which bound the intersecting edges. If the dimension of the intersection is zero, it is obvious that the intersection has no common edges. Hence, it must have common one-dimensional vertices. In summary, we can pose the following intersection matrix for discrete crisp regions:

$$\begin{array}{cc}
 & \begin{array}{ccc} \partial G & G^\circ & G^- \end{array} \\
\begin{array}{c} \partial F \\ F^\circ \\ F^- \end{array} &
\begin{pmatrix}
\mathcal{V}(\partial F) \cap \mathcal{V}(\partial G) & \mathcal{E}(\partial F) \cap \mathcal{E}(G^\circ) & \mathcal{E}(\partial F) \cap \mathcal{E}(G^-) \\
\mathcal{E}(F^\circ) \cap \mathcal{E}(\partial G) & \mathcal{C}(F^\circ) \cap \mathcal{C}(G^\circ) & \mathcal{C}(F^\circ) \cap \mathcal{C}(G^-) \\
\mathcal{E}(F^-) \cap \mathcal{E}(\partial G) & \mathcal{C}(F^-) \cap \mathcal{C}(G^\circ) & \mathcal{C}(F^-) \cap \mathcal{C}(G^-)
\end{pmatrix}
\end{array}$$

This intersection matrix can be the basis for a treatment of topological relationships between discrete *fuzzy* regions. The restriction to objects of type *region'* also leads to a restriction of fuzzy regions in the sense that we can here only permit simple fuzzy regions. A *simple fuzzy region* is a region that is α-connected and where each α-level region (see definition of an α-set in Section 4.1, (Schneider, 1999)) is an element of type *region'*. A fuzzy region is called α-*connected* if all its α-level regions F_{α_i} for $\alpha_i \in \Lambda$ are simple crisp regions, that is, elements of *region'*. Since $\alpha_i < \alpha_{i+1}$, we obtain $F_{\alpha_i} \supseteq F_{\alpha_{i+1}}$

[3] Hence, we can omit the index f in $C_f(F), E_f(F), V_f(F)$ in the following.

for $1 \leq i < n$, that is, the α-level regions are nested. This describes some kind of a "concentric" model of a fuzzy region with its core in the center and more vague parts in the core's environment. With increasing distance from the center the degree of membership decreases.

The remaining question now is how to employ the α-level regions for determining the topological relationships between two simple fuzzy regions. We use the concept of basic probability assignment (Dubois *et al.*, 1987) for this purpose. A *basic probability assignment* $m(F_{\alpha_i})$ can be attached to each α-level region F_{α_i} and can be interpreted as the probability that F_{α_i} is the "true" representative of F. It is defined as $m(F_{\alpha_i}) = \alpha_i - \alpha_{i-1}$ for $1 < i \leq n$ with $\alpha_1 = 0$ and $\alpha_n = 1$. It is easy to see that $\sum_{1<i\leq n} m(F_{\alpha_i}) = \alpha_n - \alpha_1 = 1 - 0 = 1$.

Let $\tau_p(F,G)$ be the value that represents a relationship p between two simple fuzzy regions F and G. Based on the work in (Dubois *et al.*, 1987) the relationship p between F and G can be determined by

$$\tau_p(F,G) = \sum_{i=2}^{n} \sum_{j=2}^{n} m(F_{\alpha_i}) \cdot m(G_{\alpha_j}) \cdot \tau_p(F_{\alpha_i}, G_{\alpha_j})$$

where $\tau_p(F_{\alpha_i}, G_{\alpha_j}) \in \{0,1\}$ checks the validity of predicate p for two simple crisp α-level regions F_{α_i} and G_{α_j}. This formula is equivalent to:

$$\tau_p(F,G) = \sum_{i=2}^{n} \sum_{j=2}^{n} (\alpha_i - \alpha_{i-1}) \cdot (\alpha_j - \alpha_{j-1}) \cdot \tau_p(F_{\alpha_i}, G_{\alpha_j})$$

If p is one of our eight topological predicates out of $\{$*disjoint*, *meet*, *overlap*, *equal*, *inside*, *contains*, *covers*, *coveredBy*$\}$, we can compute the degree of the corresponding topological relationship between two simple fuzzy regions, that is, $0 \leq \tau_p(F,G) \leq 1$.

7 CONCLUSIONS

This paper lays the conceptual and formal foundation for the treatment of regions blurred by the feature of fuzziness and defined over a discrete geometric domain. The belief that simply finite point sets are sufficient to appropriately model discrete regions has turned out to be a fallacy. Grid partitions are an appropriate geometric domain for discrete crisp and fuzzy spatial objects since they distinguish different components of space, each component having different dimension. The embedding of grid partitions into the Euclidean space as a topological space enables us to reason about topological relationships between discrete fuzzy regions with the aid of the 9-intersection model.

In the future the goal is to develop a finite resolution fuzzy spatial algebra including also fuzzy points and fuzzy lines and also a comprehensive set of operations. Another issue will be how to implement this type system.

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