Average Fill Rate And Horizon Length

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<u>Abstract</u>

Given a sequence of iid demands and an order up to replenishment policy with negligible lead time, we prove that average fill rate is monotonically decreasing in the number of periods in the planning horizon. This was conjectured to be true in a recent issue of this journal.

Keywords: Inventory theory, finite horizon, fill rate.

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1. Introduction

Consider a simple inventory system for a single product over a finite horizon. Demand is represented by a sequence of positive valued iid random variables. There are as many terms in the sequence as there are periods in the horizon. The replenishment system is of the order up to type. In a recent issue of this journal, Chen, Lin and Thomas [1] proved the following result for the system just described: *expected fill rate over a finite horizon with two or more periods is smaller than expected fill rate over a single period and greater than expected fill rate over an infinite horizon, assuming negligible lead time.* As pointed out by the authors in [1], this result has the interesting implication that the customary formula

$$\text{Fill Rate} = \frac{\text{Average Number of Units of Demand Filled}}{\text{Average Demand}}$$

which applies exactly to periodic review inventory systems over an infinite horizon, underestimates the fill rate achieved over a finite horizon.

It is conjectured in [1] that for a fixed order up to level, expected fill rate over a finite horizon is a monotonically decreasing function of the number of periods in the horizon; in this paper, we prove the conjecture. Incidentally, all the results about mean fill rate obtained by the authors in [1] follow from the proof of the conjecture (Theorem 2 of the present paper).

2. Results and Discussion

Let X_1, X_2, …. ,X_i, … denote an iid sequence of positive valued demand random variables. Let s, a fixed positive number, denote the order up to level. Then, provided that the replenishment lead time is zero, the number of units of demand satisfied in period i is $\text{Min}[X_i,s]$. We write $Y_i =$

Min[X_i, s]. Then expected fill rate over k periods is $E[\dfrac{Y_1 + ... + Y_k}{X_1 + ... + X_k}]$. Chen, Lin and Thomas

proved the following theorem:

Theorem 1

Let i be any positive integer and let k be any positive integer greater than 1. Then

$$E[\frac{Y_i}{X_i}] \geq E[\frac{Y_1 + ... + Y_k}{X_1 + ... + X_k}] \geq \lim_{j \to \infty} E[\frac{Y_1 + ... + Y_j}{X_1 + ... + X_j}].$$

We prove a stronger result, which was conjectured to be true by the authors of Theorem 1:

Theorem 2

$$E[\frac{Y_1 + ... + Y_k}{X_1 + ... + X_k}] \text{ is non-increasing in k.}$$

Notice that Theorem 1 can be deduced immediately from Theorem 2. Further, Theorem 2

validates the tacit assumption made by Chen, Lin and Thomas that the sequence

$u_k = E[\dfrac{Y_1 + ... + Y_k}{X_1 + ... + X_k}]$ has the property that *lim inf* u_k = *lim sup* u_k (that is, the sequence has no limit

points other than the unique limit). It follows from Theorem 2 that for a fixed demand distribution

and an order up to replenishment policy with negligible lead time, the conventional formula for fill

rate is a progressively better approximation to the actual mean fill rate for finite horizon inventory

systems spanning a greater number of periods.

The related inequality $E[\dfrac{k}{X_1 + ... + X_k}] \geq E[\dfrac{k+1}{X_1 + ... + X_{k+1}}]$ follows from the fact that if

$X_1, ..., X_k, X_{k+1}$ are positive valued iid random variables, then $\dfrac{X_1 + ... + X_k}{k}$ is greater than

$\dfrac{X_1 + ... + X_k + X_{k+1}}{k+1}$ in the convex order ([3], Theorem 2.A.12). It is interesting to note that

Theorem 2 asserts that the numbers k and k+1 in the numerators of the above inequality can be

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replaced by the random variables $Y_1 + ... + Y_k$ and $Y_1 + ... + Y_k + Y_{k+1}$ respectively, despite the

stochastic dependence between Y_i and X_i in the operand of the expectation operator. It is natural to

enquire whether Theorem 2 can be extended to a class of functions $h(X_i)$ that includes $Min(X_i, s)$

as a special case. The class of increasing concave functions would seem to be a promising

candidate but it is ruled out by the following counterexample: if $Y_i = X_i - s$ ($s > 0$), then

$$E[\frac{Y_1 + ... + Y_k}{X_1 + ... + X_k}] \le E[\frac{Y_1 + ... + Y_{k+1}}{X_1 + ... + X_{k+1}}]$$ (this follows from the convex ordering result alluded to, and

a little algebraic manipulation).

3. Proof of Theorem 2

We prove the theorem for arbitrary positive valued discrete random variables with finite support

(that is, for positive valued simple random variables). The result extends to arbitrary positive

valued random variables by convergence: for every positive valued random variable X, there

exists a sequence of positive valued simple random variables converging to X pointwise. Let s be

a fixed strictly positive real number. Let the underlying distribution consist of the points $a_1, ..., a_u$

and $b_1, ..., b_v$ where $0 \le a_1 < a_2 < ... < a_u < s < b_1 < b_2 < ..... < b_v$. Further, suppose the distribution

attaches probability p_i to the point a_i ($i = 1, ..., u$) and probability q_i to the point b_i ($i = 1, ..., v$), so

that $\sum_{i=1}^{u} p_i + \sum_{i=1}^{v} q_i = 1$. Suppose $X_1, ..., X_N$ (where $N \ge 2$) are independent random variables drawn

from this distribution and let $Y_i = Min[X_i, s]$. Then we have

$$E[\frac{Y_1 + ... + Y_N}{X_1 + ... + X_N}] = \sum \frac{N!}{k_1! k_2! ... k_u! l_1! l_2! ... l_v!} \frac{\sum_{i=1}^{u} k_i a_i + \sum_{i=1}^{v} l_i s}{\sum_{i=1}^{u} k_i a_i + \sum_{i=1}^{v} l_i b_i} \prod_{i=1}^{u} p_i^{k_i} \prod_{i=1}^{v} q_i^{l_i}$$ (1)

where the leading summation extends over all non-negative integers $k_1,\ldots,k_u, l_1,\ldots,l_v$ such that

$\sum_{i=1}^{u} k_i + \sum_{i=1}^{v} l_i = N$. Note that in forming the convolution $X_1+ \ldots + X_N$, the point a_1 occurs k_1 times,

a_2 occurs k_2 times, …, a_u occurs k_u times, b_1 occurs l_1 times, b_2 occurs l_2 times, …, b_v occurs l_v

times. Let us compare $X_1+ \ldots + X_{N-1}$ and $X_1+ \ldots + X_N$ by conditioning on the outcome of the N-th draw from the distribution in a random sample of size N. The N-th draw results in the point a_i

with probability p_i ($i = 1$ to u) and in the point b_i with probability q_i ($i = 1$ to v). If the outcome of

the N-th draw is a_1, then in forming the convolution $X_1+ \ldots + X_{N-1}$ the point a_1 occurs k_1-1 times,

a_2 occurs k_2 times, …, a_u occurs k_u times, b_1 occurs l_1 times, b_2 occurs l_2 times, …, b_v occurs l_v

times. This pattern extends in an obvious way to each of the other outcomes of the N-th draw.

Hence we have $E[\dfrac{Y_1 + \ldots + Y_{N-1}}{X_1 + \ldots + X_{N-1}}] =$

$$p_1 \sum_{\substack{k_1 \geq 1 \\ k_2,\ldots,k_u \geq 0 \\ l_1,\ldots,l_v \geq 0}} \frac{(N-1)!}{(k_1-1)! k_2! \ldots k_u! l_1! l_2! \ldots l_v!} \cdot \frac{\sum_{i=1}^{u} k_i a_i + \sum_{i=1}^{v} l_i s - a_1}{\sum_{i=1}^{u} k_i a_i + \sum_{i=1}^{v} l_i b_i - a_1} \, p_1^{k_1-1} \prod_{i=2}^{u} p_i^{k_i} \prod_{i=1}^{v} q_i^{l_i}$$

$+ \ldots$

$$+ q_v \sum_{\substack{l_v \geq 1 \\ k_1,\ldots,k_u \geq 0 \\ l_1,\ldots,l_{v-1} \geq 0}} \frac{(N-1)!}{k_1! k_2! \ldots k_u! l_1! l_2! \ldots (l_v-1)!} \cdot \frac{\sum_{i=1}^{u} k_i a_i + \sum_{i=1}^{v} l_i s - s}{\sum_{i=1}^{u} k_i a_i + \sum_{i=1}^{v} l_i b_i - b_v} \, q_v^{l_v-1} \prod_{i=1}^{u} p_i^{k_i} \prod_{i=1}^{v-1} q_i^{l_i} \qquad (2)$$

where $k_1,\ldots,k_u$, $l_1,\ldots,l_v$ are integers such that $\sum_{i=1}^{u} k_i + \sum_{i=1}^{v} l_i = N$. However, we shall soon see that it

will be useful to treat each of the (k_u+l_v) leading summations in (2) as extending over all non-

negative integers (including zero) $k_1,\ldots,k_u$, $l_1,\ldots,l_v$ such that $\sum_{i=1}^{u} k_i + \sum_{i=1}^{v} l_i = N$. Note that when

$$k_1 = 0 \text{ in } p_1 \sum \frac{(N-1)!}{(k_1-1)!k_2!..k_u!l_1!l_2!..l_v!} \frac{\sum_{i=1}^{u} k_i a_i + \sum_{i=1}^{v} l_i s - a_1}{\sum_{i=1}^{u} k_i a_i + \sum_{i=1}^{v} l_i b_i - a_1} p_1^{k_1-1} \prod_{i=2}^{u} p_i^{k_i} \prod_{i=1}^{v} q_i^{l_i} \text{ , the undefined}$$

quantity (-1)! arises; the same thing happens *mutatis mutandis* with $k_i = 0$ for every $i \geq 2$ and with

$l_j = 0$ for every $j \geq 1$. We define (-1)! to be ∞. Now using the fact that $\frac{1}{\infty} = 0$ in the extended real

number system, every term with (-1)! simply drops out when we extend each of the (k_u+l_v) leading

summations in (2) to all non-negative integers that partition N. We now have a one-to-one

correspondence between the terms in (1) and the terms in (2), since the leading summations in (1)

and in (2) extend over the same range.

     Now we need to prove that (2) when summed over all non-negative integers $k_1,\ldots,k_u$,

$l_1,\ldots,l_v$ such that $\sum_{i=1}^{u} k_i + \sum_{i=1}^{v} l_i = N$ is greater than or equal to (1) when summed over all non-

negative integers $k_1,\ldots,k_u$, $l_1,\ldots,l_v$ such that $\sum_{i=1}^{u} k_i + \sum_{i=1}^{v} l_i = N$. That is, we need to prove that (2),

summed over all partitions $k_1+\ldots+k_u+ l_1,\ldots+l_v$ of N, is greater than or equal to (1) summed over

all partitions $k_1+\ldots+k_u+ l_1,\ldots+l_v$ of N. In fact, we shall prove that given *any* partition $k_1+\ldots+k_u+$

$l_1,\ldots+l_v$ of N,

$$\frac{(N-1)!}{(k_1-1)!k_2!...k_u!l_1!...l_v!} \frac{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_is - a_1}{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_ib_i - a_1} \prod_{i=i}^{u}p_i^{k_i} \prod_{i=1}^{v}q_i^{l_i}$$

$$+ \ldots$$

$$+ \frac{(N-1)!}{k_1!k_2!...k_u!l_1!...(l_v-1)!} \frac{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_is - s}{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_ib_i - b_v} \prod_{i=1}^{u}p_i^{k_i} \prod_{i=1}^{v}q_i^{l_i}$$

$$\geq$$

$$\frac{N!}{k_1!...k_u!l_1!...l_v!} \frac{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_is}{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_ib_i} \prod_{i=1}^{u}p_i^{k_i} \prod_{i=1}^{v}q_i^{l_i}$$

which implies our claim. Let us write $\sum_{i=1}^{u}k_i = n$ and $\sum_{i=1}^{v}l_i = m$. Note that $\frac{k_i!}{(k_i-1)!} = k_i$ whenever

$k_i \geq 1$ and $\frac{k_i!}{(k_i-1)!} = 0$ when $k_i = 0$ (since $\frac{1}{\infty} = 0$). Hence for all $k_i \geq 0$, we have $\frac{k_i!}{(k_i-1)!} = k_i$.

Using this fact, a little simplification shows that we need to prove

$$(\frac{k_1}{n+m}) \frac{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_is - a_1}{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_ib_i - a_1} + \ldots + (\frac{k_u}{n+m}) \frac{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_is - a_u}{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_ib_i - a_u}$$

$$+ (\frac{l_1}{n+m}) \frac{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_is - s}{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_ib_i - b_1} + \ldots + (\frac{l_v}{n+m}) \frac{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_is - s}{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_ib_i - b_v}$$

$$\geq \frac{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_is}{\sum_{i=1}^{u}k_ia_i + \sum_{i=1}^{v}l_ib_i} \qquad (3)$$

We shall prove inequality (3) via two lemmas. Define b = $\dfrac{\sum\limits_{i=1}^{v} l_i b_i}{\sum\limits_{i=1}^{v} l_i}$ (so that $\sum\limits_{i=1}^{v} b_i l_i = \sum\limits_{i=1}^{v} l_i b$).

Lemma 1

The left hand side of inequality (3) is greater than or equal to

$$(\frac{k_1}{n+m})\frac{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i s - a_1}{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i b - a_1} + ... + (\frac{k_u}{n+m})\frac{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i s - a_u}{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i b - a_u} + (\frac{m}{n+m})\frac{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i s - s}{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i b - b}$$

Lemma 2

The right hand side of the inequality in Lemma 1 is greater than or equal to

$$\frac{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i s}{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i b}.$$

Theorem 2 follows from Lemma 1 and Lemma 2 since $\sum\limits_{i=1}^{v} b_i l_i = \sum\limits_{i=1}^{v} l_i b$. We proceed to prove the

lemmas.

Proof of Lemma 1

Note that since $\sum\limits_{i=1}^{v} b_i l_i = \sum\limits_{i=1}^{v} l_i b$, the first u terms on both sides of the inequality in Lemma 1 are

equal and the inequality simplifies to

$$(\frac{l_1}{m})\frac{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i s - s}{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i b_i - b_1} + ... + (\frac{l_v}{m})\frac{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i s - s}{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i b_i - b_v} \geq \frac{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i s - s}{\sum\limits_{i=1}^{u} k_i a_i + \sum\limits_{i=1}^{v} l_i b - b} \qquad (4)$$

Write $A = \sum_{i=1}^{u} a_i k_i$. To prove (4), we first show that the function $\Omega(x_1,...,x_v) = \dfrac{A + \sum\limits_{i=1}^{v} s x_i}{A + \sum\limits_{i=1}^{v} b_i x_i}$ defined

for non-negative x_i and restricted to the hyperplane $\sum\limits_{i=1}^{v} x_i = m - 1$ is convex in $(x_1, \ldots, x_v)$. To

show this, we write $x_i = \alpha_i + t\beta_i$ where α_i and β_i are parameters and $0 \le t \le 1$ and show that

$\Omega(\alpha_1 + t\beta_1,...,\alpha_v + t\beta_v)$ is a convex function of t. (For details about the validity of this procedure,

see [2], page 446). Note that $\sum\limits_{i=1}^{v} x_i = m - 1$ implies that $\sum\limits_{i=1}^{v} \alpha_i = m - 1$ and $\sum\limits_{i=1}^{v} \beta_i = 0$.

Hence we have

$$\Omega(\alpha_1 + t\beta_1,...,\alpha_v + t\beta_v) = \frac{A + \sum\limits_{i=1}^{v} s\alpha_i + s\sum\limits_{i=1}^{v}\beta_i}{A + \sum\limits_{i=1}^{v} b_i\alpha_i + t\sum\limits_{i=1}^{v} b_i\beta_i} = \frac{A + \sum\limits_{i=1}^{v} s\alpha_i}{A + \sum\limits_{i=1}^{v} b_i\alpha_i + t\sum\limits_{i=1}^{v} b_i\beta_i}.$$

The second derivative of this function with respect to t is

$$\frac{2(A + \sum\limits_{i=1}^{v} s\alpha_i)(\sum\limits_{i=1}^{v} b_i\beta_i)^2}{(A + \sum\limits_{i=1}^{v} b_i\alpha_i + t\sum\limits_{i=1}^{v} b_i\beta_i)^3} = \frac{2(A + \sum\limits_{i=1}^{v} s\alpha_i)(\sum\limits_{i=1}^{v} b_i\beta_i)^2}{(A + \sum\limits_{i=1}^{v} b_i(\alpha_i + t\beta_i))^3} \ge 0 \text{ since } A > 0, s\sum\limits_{i=1}^{v}\alpha_i > 0,$$

$b_i > 0$ and $x_i = \alpha_i + t\beta_i \ge 0$ by hypothesis. Hence, $\Omega(x_1,...,x_v)$ is convex. Next, we shall use the

convexity of $\Omega(x_1,...,x_v)$ to prove (4). Note that $\dfrac{l_1}{m} + ... + \dfrac{l_v}{m} = 1$. We define the following v-

vectors: $z_1 = (l_1 - 1, l_2,...,l_v), z_2 = (l_1, l_2 - 1,...,l_v),..., z_v = (l_1, l_2,...,l_v - 1)$. Now the left hand side of

(4) can be written as $(\dfrac{l_1}{m})\Omega(z_1) + (\dfrac{l_2}{m})\Omega(z_2) + ... + (\dfrac{l_v}{m})\Omega(z_v)$.

A little algebra shows that $(\frac{l_1}{m})z_1 + (\frac{l_2}{m})z_2 + ... + (\frac{l_v}{m})z_v = (l_1(1-\frac{1}{m}), l_2(1-\frac{1}{m}), ..., l_v(1-\frac{1}{m}))$.

Finally, note that $\Omega(l_1(1-\frac{1}{m}), ..., l_v(1-\frac{1}{m}))$ reduces to the right hand side of (4). Since

$\Omega(x_1, ..., x_v)$ is convex, Lemma 1 follows.

Proof of Lemma 2

We can rewrite the inequality to be proved as follows:

$$(\frac{k_1}{n+m})[1 - \frac{(b-s)m}{\sum_{i=1}^{u} k_i a_i + mb - a_1}] + ... + (\frac{k_u}{n+m})[1 - \frac{(b-s)m}{\sum_{i=1}^{u} k_i a_i + mb - a_u}] + (\frac{m}{n+m})[1 - \frac{(b-s)(m-1)}{\sum_{i=1}^{u} k_i a_i + (m-1)b}]$$

$$\geq [1 - \frac{(b-s)m}{\sum_{i=1}^{u} k_i a_i + mb}].$$

Since $b \geq s$, this is equivalent to

$$(\frac{k_1}{n+m})[\frac{m}{\sum_{i=1}^{u} k_i a_i + mb - a_1}] + ... + (\frac{k_u}{n+m})[\frac{m}{\sum_{i=1}^{u} k_i a_i + mb - a_u}] + (\frac{m}{n+m})[\frac{(m-1)}{\sum_{i=1}^{u} k_i a_i + (m-1)b}]$$

$$\leq \frac{m}{\sum_{i=1}^{u} k_i a_i + mb}$$

$$\Longleftrightarrow$$

$$\frac{k_1}{\sum_{i=1}^{u} k_i a_i + mb - a_1} + ... + \frac{k_u}{\sum_{i=1}^{u} k_i a_i + mb - a_u} + \frac{(m-1)}{\sum_{i=1}^{u} k_i a_i + (m-1)b} \leq \frac{n+m}{\sum_{i=1}^{u} k_i a_i + mb}$$

Note that

$$\frac{k_j + \frac{k_j a_j}{\sum_{i=1}^{u} k_i a_i + mb - a_j}}{\sum_{i=1}^{u} k_i a_i + mb} = \frac{k_j}{\sum_{i=1}^{u} k_i a_i + mb - a_j} \quad (j = 1, 2, ..., u)$$

Also,
$$\frac{(m-1)+\dfrac{(m-1)b}{\sum_{i=1}^{u}k_ia_i+(m-1)b}}{\sum_{i=1}^{u}k_ia_i+mb}=\frac{m-1}{\sum_{i=1}^{u}k_ia_i+(m-1)b}.$$

So we need to show that

$$\frac{k_1+\dfrac{k_1a_1}{\sum_{i=1}^{u}k_ia_i+mb-a_1}}{\sum_{i=1}^{u}k_ia_i+mb}+...+\frac{k_u+\dfrac{k_ua_u}{\sum_{i=1}^{u}k_ia_i+mb-a_u}}{\sum_{i=1}^{u}k_ia_i+mb}+\frac{(m-1)+\dfrac{(m-1)b}{\sum_{i=1}^{u}k_ia_i+(m-1)b}}{\sum_{i=1}^{u}k_ia_i+mb}\le\frac{n+m}{\sum_{i=1}^{u}k_ia_i+mb}$$

$$\Longleftrightarrow$$

$$\frac{k_1a_1}{\sum_{i=1}^{u}k_ia_i+mb-a_1}+...+\frac{k_ua_u}{\sum_{i=1}^{u}k_ia_i+mb-a_u}+\frac{(m-1)b}{\sum_{i=1}^{u}k_ia_i+(m-1)b}\le 1$$

the preceding equivalence following from the fact that $\sum_{i=1}^{u}k_i=n$.

Now the left hand side of the inequality above can be written as

$$\frac{k_1a_1}{\sum_{i=1}^{u}k_ia_i+(m-1)b+(b-a_1)}+...+\frac{k_ua_u}{\sum_{i=1}^{u}k_ia_i+(m-1)b+(b-a_1)}+\frac{(m-1)b}{\sum_{i=1}^{u}k_ia_i+(m-1)b}$$

$$\le\frac{k_1a_1}{\sum_{i=1}^{u}k_ia_i+(m-1)b}+...+\frac{k_ua_u}{\sum_{i=1}^{u}k_ia_i+(m-1)b}+\frac{(m-1)b}{\sum_{i=1}^{u}k_ia_i+(m-1)b}=1$$

The last inequality follows from $b-a_i\ge 0$ for $i=1,2,\dots,u$. This proves Lemma 2.

Lemma 1 and Lemma 2 together yield Theorem 2, completing our proof.

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