Dynamics of 2-Worker Bucket Brigade Assembly Line with Blocking and Instantaneous Walk-Back

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Abstract

We analyze the dynamics of 2-worker m-stations bucket brigade assembly lines where the velocities of the workers on the stations are arbitrary, albeit fixed constants over each station. We provide a complete characterization of the dynamics under *blocking* and *instantaneous* walk-back.

Key words: Production/Scheduling, Bucket brigade lines, Analysis of dynamical systems

1. Introduction

The standard model of the bucket brigade assembly line [3] for the n workers m stations case ($n \ll m$) considers an assembly line that is partitioned into m stations, each station corresponding to a subtask of the total work content. A job has to be processed at all m stations, in sequence, to be completed. The workers are ordered 1 to n, upstream to downstream, and this order is maintained across stations at all times. Each worker picks a job and processes it on a station with a velocity commensurate with his skill at that station. The worker then takes the job to the next station to continue processing it. In the *blocking* model, two workers are not allowed to occupy the same station simultaneously. The downstream worker has precedence over the upstream worker in the sense that the upstream worker has to wait until the downstream worker has released the station. When a worker arrives at a station that is busy, he is considered blocked on that station and he does not seek any work until his successor leaves that station. When the last worker completes processing his job, all workers *simultaneously* hand off their jobs in their current states to their respective successors, picking up the job of their respective predecessors; the first worker starts processing a new job. In the *instantaneous* walk-back model, the entire set of hand-offs happens instantaneously.

In this note we consider the 2-worker m-stations bucket brigade assembly line with blocking and instantaneous walk back. We provide a complete characterization of the

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dynamics under the model where the workers can have different velocities on different stations, albeit constant velocities over each station. The assembly line is represented by the interval $I = [0, 1]$. The processing of a job begins at 0 and ends at 1. The station S_i is represented by the interval $[P(i), P(i+1))$, with $P(1) = 0, P(m+1) = 1$. The upstream worker is denoted by W_1 and the downstream worker by W_2.

2. Prior Work and Contributions Made

Bartholdi *et al.* [3] analyzed the n workers m stations case with blocking and instantaneous walk-back, for the *special case* where the workers are sequenced slowest to fastest. They showed that if the workers can be sequenced such that each is faster than her predecessor at all stations, then there is a unique stable fixed point to which the system converges independent of the starting positions of the workers. They then studied the 2 and 3 workers case [2] under the assumption that each worker has a *constant velocity* over the entire assembly line with the work content distributed uniformly over the entire assembly line (hence, the concept of stations does not exist). In this framework, if workers can be sequenced from slowest to fastest, they can *never* be blocked. Furthermore, the production rate under such conditions is the sum of the velocities of the workers and is the maximum achievable across all sequencing of the workers [3].

Armbruster *et al.* [1] studied the dynamics of the 2 workers case where W_1 is faster than W_2 in the interval $[0, X)$ and slower in the interval $[X, 1]$. They considered both the cases where W_1 is allowed to pass over W_2 and where W_1 can be blocked by W_2. Although not as restrictive as the assumption that one worker's speed dominate the other uniformly, this framework can not model the general case where W_1 is faster/slower than W_2 on an arbitrary, not necessarily contiguous set of stations.

In this note we generalize the results reported in [1]. We fully characterize the mapping f that specifies the successive hand-off locations between W_1 and W_2 in Section 3. After demonstrating that f has a unique fixed point and can have no periodic cycles of period > 2, we show how to algorithmically compute the fixed point and the critical point (defined in Section 3), in Section 4. In Section 5, we determine the necessary and sufficient conditions for the global stability of the fixed point, and show how to algorithmically ascertain this in Section 6. In Section 7 we analyze throughput, and in Section 8, we present concluding remarks.

3. Characterization of the mapping function

We begin by characterizing the mapping f that specifies the successive hand-off locations between W_1 and W_2. Specifically, if W_1 begins at the start of the assembly line (i.e., at 0) and W_2 begins at $x \in [0, 1]$, then $f(x)$ denotes the position of W_1 at the time when W_2 reaches the end of the assembly line (i.e., 1). Naturally, after hand-off, W_1 begins at 0 and W_2 at $f(x)$. We characterize f through the following set of theorems.

Theorem 3.1. f *is continuous and piece-wise linear.*

Proof. Let V_{1max} and V_{2min} be, respectively, the maximum and minimum velocities of W_1 and W_2 over the entire assembly line. Let $x_0 \in [0, 1]$ be given. Consider $x \in [0, 1]$ such that $| x_0 - x |< \delta$ for small δ. When W_2 starts at x instead of x_0, the amount of time gained or lost by W_1 when W_2 reaches 1 is $\Delta t < \frac{\delta}{V_{2min}}$. Therefore $| f(x_0) - f(x) | \leq \Delta t * V_{1max}$. For a given $\epsilon > 0$ one can choose a δ such that $\Delta t * V_{1max} < \frac{\delta * V_{1max}}{V_{2min}} < \epsilon$. Hence if $| x_0 - x | < \delta$ then $| f(x_0) - f(x) | < \epsilon$ which proves the continuity of f at x_0. To prove that f is piece-wise linear, let x_0 and $f(x_0)$ lie in the interior of their respective stations, i.e., not at their station's boundaries. Let V_1 and V_2 be, respectively, the velocities of W_1 at $f(x_0)$ and W_2 at x_0. Let $x = x_0 \pm \Delta x$. Then $f(x) = f(x_0) \mp \frac{\Delta x * V_1}{V_2}$, i.e., f is *linear* in the neighborhood of x_0. □

Definition W_1 is said to be *blocked* on x, if for W_2 beginning at x (and W_1 at 0) there exists a station S_i such that W_1 is blocked at S_i, i.e., W_1 reaches S_i before W_2 leaves S_i.

Theorem 3.2. *f is non-increasing. It is constant up to a point $\tilde{x}$ beyond which it is strictly decreasing.*

Proof. If W_1 is not blocked on x, then $\forall y > x$, W_1 is not blocked on y. If W_1 is blocked on x at a station S_i, then $\forall y < x$, W_1 will be blocked at S_i and hence $f(y) = f(x)$ in this range. Therefore if W_1 is blocked $\forall x < \tilde{x}$ and not blocked on $\tilde{x}$, we see that $\forall x < \tilde{x}, f(x) = f(\tilde{x})$. Finally, let t_x be the total time for which W_1 processes a job when W_2 begins at x. Then for $\tilde{x} < x < y$, $t_{\tilde{x}} > t_x > t_y$, and therefore, $f(\tilde{x}) > f(x) > f(y)$. In other words, f is *strictly decreasing* after $\tilde{x}$. □

We label $\tilde{x}$ the *critical point.* Figure 1 presents an example of f.

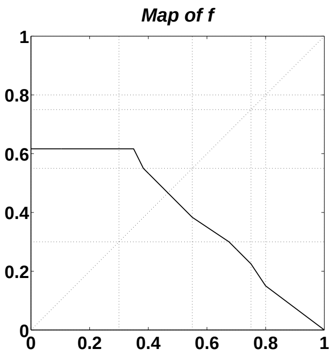

Figure 1: Map of f. Horizontal and vertical lines denote station boundaries. $\tilde{x}$ is the point where the graph changes from constant to strictly decreasing. x_0 is the point where the graph intersects the diagonal.

Theorem 3.3. *f has a unique fixed point and has no periodic cycles of period > 2.*

Proof. It follows from the Brouwer's fixed point theorem that f has a fixed point. Since f is non-increasing this fixed point is *unique.* Moreover, it is well known from dynamical systems theory that a monotonically non-increasing function (in our case, the mapping f) cannot have periodic cycles of period > 2. We provide a proof here for completeness.

Let $x_1, x_2, \ldots, x_n$ form a cycle of period $n > 2$ with $f(x_i) = x_{i+1}, f(x_n) = x_1$. Without loss of generality let $x_1 < x_i, \forall i \neq 1$. Since $x_2 > x_1$, it follows from Theorem 3.2 that $f(x_2) = x_3 < f(x_1) = x_2$, i.e., $x_1 < x_3 < x_2$, and hence $x_1 < x_3 < x_4 < x_2$, and hence $x_1 < x_3 < x_5 < x_4 < x_2$. It follows that the x_i's are ordered as $x_1 < x_3 < x_5 < \ldots < x_6 < x_4 < x_2$. Importantly $x_1 < x_n < x_2$. Therefore $f(x_n) = x_1 > f(x_2) = x_3$, leading to a contradiction. □

4. Algorithmic computation of the critical point and the fixed point

4.1. Critical point ($\tilde{x}$)

Let $x = \tilde{x} - \Delta x$, where Δx is an arbitrarily small positive number. Let S_p be the last station at which W_1 is blocked when W_2 begins at x. Hence when W_2 begins at $\tilde{x}$, W_1 will enter S_p and W_2 will leave S_p *simultaneously.* This property can be used to compute the value of $\tilde{x}$.

Let the final station be $S_m = [P(m), P(m+1) = 1)$. Allow W_2 to begin at positions $P(m), P(m-1), \ldots$, till the first position $P(k)$ is found on which W_1 is blocked (i.e., W_1 beginning at $P(1) = 0$ is blocked when W_2 begins at $P(k)$). Hence $P(k) \leq \tilde{x} < P(k+1)$. Let S_p be the corresponding last station at which W_1 is blocked. Then $\tilde{x}$ is that position for which the time taken by W_2 to reach $P(p+1)$ beginning at $\tilde{x}$ equals the time taken by W_1 to reach $P(p)$ beginning at 0. Let $V_i(j)$ denote the velocity of worker i at station j. Then, $\tilde{x} = P(k+1) - V_2(k) * \left[\sum_{j=1}^{p-1} \frac{P(j+1)-P(j)}{V_1(j)} - \sum_{j=k+1}^{p} \frac{P(j+1)-P(j)}{V_2(j)}\right]$

4.2. Fixed point (x_0)

Since x_0 is a fixed point, $f(x_0) = x_0$. We consider all three possibilities: $x_0 < \tilde{x}$, $x_0 = \tilde{x}$, and, $x_0 > \tilde{x}$, and show how x_0 can be computed in each case. Let S_k denote the station in which $\tilde{x}$ occurs, i.e., $P(k) \leq \tilde{x} < P(k+1)$.

4.2.1. case 1: $x_0 = \tilde{x}$

As defined above let $V_i(j)$ denote the velocity of worker i at station j. Then if $\sum_{j=1}^{k-1} \frac{P(j+1)-P(j)}{V_1(j)} + \frac{\tilde{x}-P(k)}{V_1(k)} = \frac{P(k+1)-\tilde{x}}{V_2(k)} + \sum_{j=k+1}^{m} \frac{P(j+1)-P(j)}{V_2(j)}$ then $x_0 = \tilde{x}$.

4.2.2. case 2: $x_0 < \tilde{x}$

Lemma 4.1. *If $x_0 < \tilde{x}$ and x_0 occurs in station S_i, then S_i is the only station at which W_1 is blocked when W_2 begins at x_0.*

Proof. Since $x_0 < \tilde{x}$, W_1 is blocked on x_0. Furthermore, since W_2 begins at x_0, S_i is the first station at which W_1 can be blocked. Since x_0 is a fixed point and lies in S_i, W_1 does not reach S_{i+1} when W_2 begins at x_0. The claim then follows. □

Based on the definition of $\tilde{x}$, we see that $S_i = S_k$ and $P(k) \leq x_0 < \tilde{x} < P(k+1)$, i.e., x_0 and $\tilde{x}$ occur in the *same station* and $x_0 = P(k)$ if and only if $k = m$. If $k \neq m$, it follows from Lemma 4.1 that the time taken by W_1 to reach x_0 beginning at $P(k)$ equals the time taken by W_2 to reach the end of the assembly line beginning at $P(k+1)$. Therefore, $x_0 = P(k) + V_1(k) * \left[\sum_{j=k+1}^{m} \frac{P(j+1)-P(j)}{V_2(j)}\right]$

4.2.3. case 3: $x_0 > \tilde{x}$

Let x_0 occur in station S_i. Since $x_0 > \tilde{x}$, W_1 is not blocked. Therefore, the time taken by W_1 to reach x_0 beginning at $P(1) = 0$ equals the time taken by W_2 to reach $P(m+1) = 1$ beginning at x_0, which can be computed as,
$x_0 = \left[\frac{V_1(i)*V_2(i)}{V_1(i)+V_2(i)}\right] * \left[\sum_{j=i+1}^{m} \frac{P(j+1)-P(j)}{V_2(j)} - \sum_{j=1}^{i-1} \frac{P(j+1)-P(j)}{V_1(j)} + \frac{P(i)}{V_1(i)} + \frac{P(i+1)}{V_2(i)}\right]$

5. Necessary and sufficient conditions for a globally stable fixed point

Theorem 5.1. *If $x_0 \leq \tilde{x}$ then x_0 is globally stable.*

Proof. From Theorem 3.2, $\forall x \leq \tilde{x}, f(x) = f(\tilde{x}) = f(x_0) = x_0$. Moreover, $\forall x > \tilde{x}, f(x) < f(\tilde{x}) = x_0$ and hence $f^2(x) = x_0$. Thus x_0 is a globally stable fixed point. □

Hereafter we consider the more interesting case: $x_0 > \tilde{x}$. We first provide the necessary conditions for the stability of the fixed point and the necessary conditions to *avoid* stable period 2 cycles. From these we derive the necessary and sufficient conditions for the global stability of the fixed point.

Theorem 5.2. *x_0 is a stable fixed point if and only if*

1. *If f is differentiable at x_0 then $|f^{'}(x_0)| < 1$.*
2. *If f is not differentiable at x_0 then for $\Delta x \to 0^+, x_1 = x_0 + \Delta x, x_2 = x_0 - \Delta x$ $f^{'}(x_1) * f^{'}(x_2) < 1$*

Proof. Case (i) is a well known condition from dynamical systems theory. For case (ii) let $x_1 = x_0 + \Delta x, x_2 = x_0 - \Delta x$. It follows from Theorem 3.2 that $f^2(x_1) > x_0$. Let $\Delta z = f^2(x_1) - x_0$ and let $t = \frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} * \frac{\Delta y}{\Delta x}$, where $\Delta y = f(x_1) - x_0$. As $\Delta x \to 0^+, t = f^{'}(x_1) * f^{'}(f(x_1))$. Since f is piece-wise linear, $f^{'}(f(x_1)) = f^{'}(x_2)$. Therefore $t = f^{'}(x_1) * f^{'}(x_2)$. Since $f^{'}(x_1), f^{'}(x_2) < 0$, $f^{'}(x_1) * f^{'}(x_2) > 0$. If $f^{'}(x_1) * f^{'}(x_2) < 1$ then $\Delta z < \Delta x$. The proof for a perturbation in the other direction follows along similar lines. Conversely, if x_0 is stable then $\Delta z < \Delta x$ and hence $\frac{\Delta z}{\Delta x} = t = f^{'}(x_1) * f^{'}(x_2) < 1$. □

Theorem 5.3. *If $f^2(\tilde{x}) < \tilde{x}$ then f has a stable period 2 cycle.*

Proof. Let $f(\tilde{x}) = \tilde{y}, f(\tilde{y}) = \tilde{w}$, and $\tilde{w} < \tilde{x}$. From Theorem 3.2 $f(\tilde{w}) = \tilde{y}$, and therefore f has a period 2 cycle: $\langle \tilde{w}, \tilde{y} \rangle$. Also, the intervals $[0, \tilde{x}] \ni \tilde{w}$ and $[f^{-1}(\tilde{x}), 1] \ni \tilde{y}$ converge to the cycle in a single period. Hence, the cycle is stable. □

We also notice that if $f^2(\tilde{x}) = \tilde{x}$ then $\forall x \leq \tilde{x}$, $f^2(x) = \tilde{x}$, and therefore the interval $[0, \tilde{x}]$ converges to the period 2 cycle. Moreover, in this case it is possible that for $x > \tilde{x}, f^{2n}$ may not converge to $\tilde{x}$, i.e., f might be *structurally unstable.* Hence the necessary conditions for the global stability of the fixed point is $f^2(\tilde{x}) > \tilde{x}$.

We now provide necessary and sufficient conditions for the global stability of the fixed point.

Theorem 5.4. *x_0 is globally stable if and only if $\tilde{x} \to x_0$ under the iterates of f.*

Proof. If x_0 is globally stable, then by definition $f^n(\tilde{x}) \to x_0$. To prove the converse, let $\tilde{x} \to x_0$ and define $\tilde{y} = f(\tilde{x})$. It follows from Theorem 3.2 that $\forall x \leq \tilde{x}, f(x) = \tilde{y}$ and hence $\forall x \leq \tilde{x}, x \to x_0$. Moreover, since $\tilde{x} \to x_0$ it follows from Theorem 5.3 that $f(\tilde{y}) > \tilde{x}$. Let $\tilde{z} > \tilde{y}$ be such that $f(\tilde{z}) = \tilde{x}$. We see that $\forall z \geq \tilde{z}, f^2(z) = \tilde{y}$ and hence $\forall z \geq \tilde{z}, z \to x_0$. What remains to be shown is the convergence of the points in the interval $I = (\tilde{x}, \tilde{z})$ to x_0. Define $I_0 = (x_0, \tilde{z})$, and $I_n = \big(f^{n-1}(\tilde{x}), x_0\big)$ for $n \geq 1$, where $f^0(\tilde{x}) = \tilde{x}$. We note that $I = I_0 \cup I_1 \cup \{x_0\}$ and $f(I_n) = I_{n+1}, \forall n \geq 0$. Since the endpoint $f^{n-1}(\tilde{x})$ of I_n converges to x_0, it follows that all points in I converge to x_0. Hence $\forall x \in [0,1], x \to x_0$, making x_0 a globally stable fixed point. □

As noted earlier some period 2 cycles can assume a *dual* role both as an attractor and a repeller, i.e., their domain of attraction is one-sided with the other side being a repeller, for example, when f is *structurally unstable.* If these period 2 cycles are counted twice for their dual role, then we have the following.

Theorem 5.5. *If x_0 is a stable fixed point, then the number of period 2 cycles is even.*

Proof. Since x_0 is sandwiched between the points of any period 2 cycle, it suffices to consider the region $(x_0, 1]$. Consider the map f^2 and the diagonal line L defined by $f^2(x) = x$. Label any intersection of f^2 with L *proper*, if f^2 completely intersects L and does not merely touch L. Since x_0 is stable, for $x = x_0 + \Delta x$, infinitesimal $\Delta x > 0$, we have $f^2(x) < x$. Moreover, since W_1 never enters the last station, $\forall x, f(x) < 1$ and hence $f^2(1) < 1$. Therefore the number of proper intersections of f^2 with L in the range $(x_0, 1]$ is *even.* If f^2 merely touches L at y, then it is easy to see that the period 2 cycle involving y is structurally unstable and hence is counted twice for its dual role. Hence the number of period 2 cycles is even. □

Corollary 5.6. *If x_0 is stable and a period 2 cycle exists, then x_0 is not globally stable.*

Proof. If the period 2 cycle is a repeller, then from Theorem 5.5 it follows that an attracting period 2 cycle exists and hence there exists an interval converging to this period 2 cycle. Even in a case where this period 2 cycle exhibits dual behavior, there exists an interval converging to it. Hence the fixed point x_0 is not globally stable. □

6. Algorithmic determination of the global stability of the fixed point

We saw in the previous section that whether or not the fixed point x_0 is globally stable can be determined by considering the following exhaustive list of scenarios: (i) if $x_0 \leq \tilde{x}$ then x_0 is globally stable, (ii) if $x_0 > \tilde{x}$ and $f^2(\tilde{x}) \leq \tilde{x}$ then x_0 is not globally stable (period 2 cycle exists), and (iii) if $x_0 > \tilde{x}$ and $f(\tilde{x}) < f^{-1}(\tilde{x})$ then x_0 is globally stable iff $\tilde{x} \to x_0$ under the iterates of f.

Scenarios (i) and (ii) can be easily verified. Scenario (iii) concerns the dynamics of the points in the interval $I = (\tilde{x}, \tilde{z})$ where $\tilde{z} = f^{-1}(\tilde{x})$, since in this case $f(I) \subset I$ and furthermore $f^2([0, 1]) \subset I$. Based on the observation that worker W_1 is *not* blocked $\forall x \in I$, we can determine the location $f(x)$ for any $x \in I$ using the following procedure.

Case 1: $x > x_0$. Let t be the time taken by W_2 to reach x beginning at x_0. W_1 would then have traveled for an additional time t before hand-off, had W_2 begun at x_0. Therefore, if W_1 travels for time t from $f(x)$ he would reach $f(x_0) = x_0$, or equivalently, if W_1 begins at x_0 and travels *backward* for time t he would reach $f(x)$.

Case 2: $x < x_0$. Let t be the time taken by W_2 to reach x_0 beginning at x, or equivalently, the time taken by W_2 to reach x traveling *backward* beginning at x_0. In this case, W_1 travels for an additional time t before hand-off in comparison to when W_2 begins at x_0. Hence $f(x)$ is the position that W_1 reaches traveling for time t beginning at x_0.

f on the interval I can therefore be computed as follows. W_1 and W_2 begin at x_0 and travel in *opposite* directions. If W_2 takes time t to reach x, then $f(x)$ is the position reached by W_1 at time t.

We demonstrated in the previous section that necessary and sufficient conditions for the global stability of the fixed point x_0 is the absence of period 2 cycles. This corresponds to the criterion $\forall x > x_0\ f^2(x) < x$. Since $\forall x > \tilde{z}\ f^2(x) = f^2(\tilde{z})$, it suffices to check if $\forall x \in (x_0, \tilde{z})\ f^2(x) < x$. We give a computation procedure to determine this. For all $x \in I$ such that $x > x_0$ compute the time taken by W_1 and W_2 to reach x beginning at x_0. Label these functions f_{1f} and f_{2f}, respectively. Clearly, f_{1f} and f_{2f} are piece-wise linear and monotonically increasing with derivatives given by the inverse of the velocities of the workers on the corresponding stations. The derivatives may not exist only at station boundaries. Likewise, for all $x \in I$ such that $x < x_0$, compute the functions f_{1b} and f_{2b} as the time taken by W_1 and W_2, respectively, to reach x beginning at x_0, traveling backward. Clearly, these functions too are piece-wise linear and monotonically increasing. Define function $g = f_{1f}^{-1} \circ f_{2b} \circ f_{1b}^{-1} \circ f_{2f}$. From the description of f^2 given above it is easy to check that $f^2 = g$. Whether $g(x) < x, \forall x \in (x_0, \tilde{z})$ can be verified by plotting g. We should comment that computing f_{1f}^{-1} and f_{1b}^{-1} are straightforward since they define the distance traveled by the workers for a given time t.

7. Throughput

We have ascertained that in the case of 2-worker m-stations assembly lines, the system will either settle to the unique fixed point or to a period 2 cycle. Computing throughput (production rate) in either of these scenarios is straightforward.
Fixed Point: Let the fixed point x_0 occur at the station $S_k = [P(k), P(k+1))$. The time taken to produce one item ($T_{fixedPoint}$) is the time taken by W_2 to reach the end of the assembly line starting from x_0, which is $\frac{P(k+1)-x_0}{V_2(k)} + \sum_{j=k+1}^{m} \frac{P(j+1)-P(j)}{V_2(j)}$. The production rate is $PR_{fixedPoint} = 1/T_{fixedPoint}$.
Period 2 cycle: Let $\langle x, y \rangle$ denote a period 2 cycle such that $f(x) = y$ and $f(y) = x$. Without loss of generality, let $y < x_0 < x$. The time taken to produce 2 items is then the sum of times taken by W_2 to reach the end of the assembly line starting from x and from y. Let x and y occur at stations l and p respectively ($p < l$). Then the time taken to produce 2 items is:
$T_{period2cycle} = \frac{P(l+1)-x}{V_2(l)} + \sum_{j=l+1}^{m} \frac{P(j+1)-P(j)}{V_2(j)} + \frac{P(p+1)-y}{V_2(p)} + \sum_{j=p+1}^{m} \frac{P(j+1)-P(j)}{V_2(j)}$. The production rate is $PR_{period2cycle} = 2/T_{period2cycle}$

7.1. Comparison between production rates

Comparing the production rates between a fixed point and a period 2 cycle is also straightforward. Since $y < x_0 < x$ and hence $p < k < l$, the difference in time to produce 2 items between these scenarios is $T_{diff} = T_{period2cycle} - 2 * T_{fixedPoint}$ which can be shown to be the difference in the times taken by W_2 to reach x_0 starting from y and to reach x starting from x_0. From the description of the dynamics given in Section 6, it follows that the time taken by W_2 to reach x starting from x_0 equals the time taken by W_1 to reach x_0 starting from y. Hence, $T_{diff} = (P(p+1) - y) * \left(\frac{1}{V_2(p)} - \frac{1}{V_1(p)}\right) + \sum_{j=p+1}^{k-1} (P(j+1) - P(j)) * \left(\frac{1}{V_2(j)} - \frac{1}{V_1(j)}\right) + (x_0 - P(k)) * \left(\frac{1}{V_2(k)} - \frac{1}{V_1(k)}\right)$. If $T_{diff} > 0$, the fixed point has a higher production rate than the period 2 cycle, and vice versa. For the case considered by Armbruster *et al.* [1], the fixed point was shown to have a higher production rate than the period 2 cycle. For our more general case one can easily construct examples where the stable period 2 cycle has a higher production rate. Consider an assembly line with 8 stations, defined by the intervals $[0, 0.2)$, $[0.2, 0.25)$, $[0.25, 3)$, $[0.3, 0.45)$, $[0.45, 0.5)$, $[0.5, 0.7)$, $[0.7, 0.9)$ and $[0.9, 1)$ with the velocity of W_1 being 1 on all stations and the velocity of W_2 being $\{1, 0.9, 1.4, 1.1, 0.9, 1.1, 0.5, 3\}$ on the respective stations. Simulation shows that this assembly line has a stable fixed point at 0.5603, an unstable period 2 cycle $\langle 0.3987, 0.7173\rangle$, followed by a stable period 2 cycle $\langle 0.3667, 0.7333\rangle$. The production rate for the stable fixed point and the stable period 2 cycle are 1.7848 and 1.7966, respectively (T_{diff} is -0.0074) indicating that the stable period 2 cycle has a higher production rate than the stable fixed point.

7.2. Dependence of throughput on the velocities of the workers

At first sight one would expect the throughput to increase with an increase in the velocity of either of the workers. This however is not true as the following simple example demonstrates.

Consider a two station assembly line where W_1 is faster than W_2 in station 1. Assume, in addition, that the fixed point x_0 lies in station 1. Clearly, W_1 remains blocked until W_2 reaches station 2. When the velocity of W_2 is increased in station 2, the fixed point shifts to the left. Let the new fixed point be denoted by x_0'. Then, $x_0' < x_0$. The difference in time to produce one item between the former and the latter scenario can be shown to be equal to the difference in the times taken by W_1 and W_2 to reach x_0 from x_0'. Since W_1 is faster than W_2 in station 1, this quantity is negative implying that the former scenario has a higher throughput than the latter. In essence, the production rate drops when the velocity of W_2 is increased.

8. Conclusion

This note generalizes the results in [1]. When applied to the 2 worker case, [3] holds that x_0 is globally stable when W_2 dominates W_1 over the entire assembly line. We show that in this case the imposed condition on g defined in Section 6 is satisfied.

Let $x = x_0 + \Delta x, \Delta x > 0$. Let $g(x) = f^2(x) = x_0 + \Delta z, \Delta z > 0$. We demonstrate that if W_2 dominates W_1 then $\Delta z < \Delta x$, and hence x_0 is globally stable. Let $t_i, i \in 1, 2$ be the time taken by W_i to reach x beginning at x_0. Since W_2 dominates W_1, $t_2 < t_1$. Let W_1 reach position y traveling backward from x_0 for time t_2. Then, $f(x) = y$. Since W_2 dominates W_1, the time taken by W_2 to reach y, say t_3, is less than t_2. Hence $t_3 < t_2 < t_1$. Since Δz (respectively, Δx) is the distance covered by W_1 traveling forward from x_0 for time t_3 (respectively, t_1), $\Delta z < \Delta x$, implying that x_0 is globally stable. However, it is a simple exercise to construct examples that show that the criterion is a sufficient and not necessary condition for the global stability of the fixed point.

References

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