Homework #1
(due 2004/01/20)

January 9, 2004

1. Problem 1.7.2 from Small's book.
2. Problem 1.7.5 from Small's book.
3. The two distance metrics

$$d_{\text{Procrustes}}(\tau_1, \theta(\tau_2)) = \arccos(\sum_{k=1}^{2n} \tau_{1k}\theta(\tau_2)_k) \tag{1}$$

and

$$d_{\text{least-squares}}(\tau_1, \theta(\tau_2) = \sum_{j=1}^{n} \| \mathbf{x}_{1j} - R(\theta)\mathbf{x}_{2j} \|^2 \tag{2}$$

are to be used for pre-shapes τ_1 and $\theta(\tau_2)$ where $\theta(\tau_2)$ denotes a rotation of pre-shape τ_2 by an angle θ. The preshape $\tau_1 = \{\mathbf{x}_{11}, \mathbf{x}_{12}, \ldots, \mathbf{x}_{1n}\}$ where $\mathbf{x}_{1j} \in \mathbb{R}^2, \forall j$ and the preshape $\tau_2 = \{\mathbf{x}_{21}, \mathbf{x}_{22}, \ldots, \mathbf{x}_{2n}\}$ where $\mathbf{x}_{2j} \in \mathbb{R}^2, \forall j$. The point-sets are to be generated in the following manner. First generate $y_1 \in \mathbb{R}^{2n}$ and then generate a noisy, rotated version of y_1, namely $y_2 = \theta^*(y_1) + n$. The notation $\theta^*(y_1)$ denotes a rotation through angle θ^* of y_1 with the centroid of y_1 being the center of rotation. Once $\theta^*(y_1)$ is generated, add zero mean Gaussian noise with different standard deviations to generate different noisy y_2. The rotation matrix $R(\theta) \stackrel{\text{def}}{=} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$. After generating y_1 and y_2, compute the pre-shapes τ_1 and τ_2 by removing the location and scale.

i) Using $n = 20$, and $\theta^* = \frac{\pi}{4}$, plot the two distance metrics for different values of the Gaussian noise standard deviation and for different values of θ. For the different standard deviations used, report the values of $e_{\text{Procrustes}}(\sigma; \theta^*) = \inf_\theta d_{\text{Procrustes}}(\tau_1, \theta(\tau_2))$ and $e_{\text{least-squares}}(\sigma; \theta^*) = \inf_\theta d_{\text{least-squares}}(\tau_1, \theta(\tau_2)$. You may use a random point-set for y_1 or a point-set of your own design. Make sure to explore the noise performance by choosing standard deviations that slowly degrade the metrics. I expect to see the full spectrum ranging from almost exact shape equivalence to non-equivalence.

ii) Using a Taylor series approximation of the arccos function, demonstrate a relationship between the two metrics.

iii) What can you conclude from this empirical and analytical work?