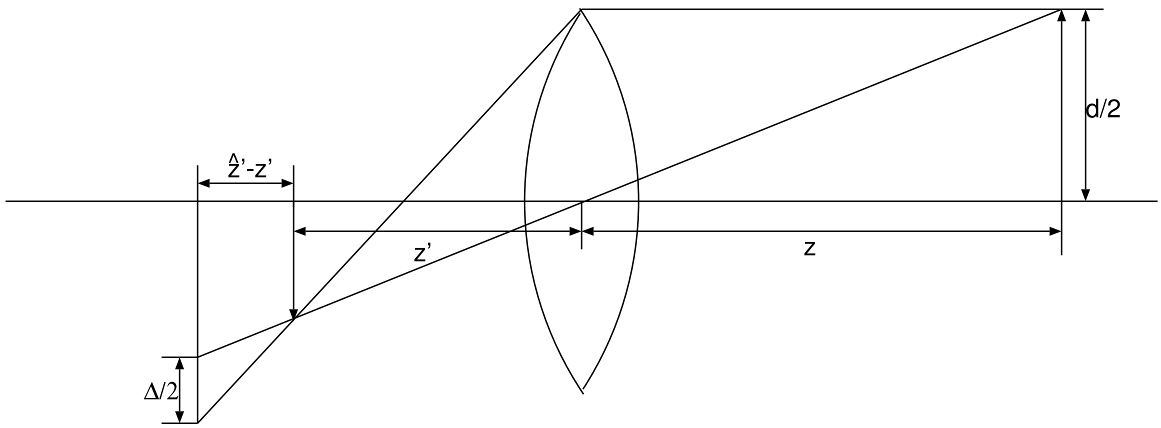
Homework #1 Solutions

1.5. The scenario is depicted in the figure below.



In the figure, $\hat{z}$ is not shown. The basic idea is that a blur circle is formed as the screen is moved from z' to $\hat{z}'$. From the figure, we see that we can use similar triangles to assert that

$$\frac{\Delta/2}{|\hat{z}' - z|} = \frac{d/2}{z'}$$

which gives us

$$\Delta = |\hat{z}' - z|\frac{d}{z'}$$

which is the desired result.

For the second part of the problem, assume that z_1 is the nearest point for which $z_1 > z$ and for which the blurred image occurs at z_1' such that

$$\epsilon = d\frac{z_1' - z'}{z'}.$$

Similarly, let z_2 be the farthest point for which $z > z_2$and for which the blurred image occurs at z_2' such that

$$\epsilon = d\frac{z' - z_2'}{z'}.$$

These quantities are in terms of z', $z_1'$and z_2' and can be rewritten using the thin lens equation

$$\frac{1}{z'} - \frac{1}{z} = \frac{1}{f}, \frac{1}{z_1'} - \frac{1}{z_1} = \frac{1}{f}, \text{ and } \frac{1}{z_2'} - \frac{1}{z_2} = \frac{1}{f}.$$

The corresponding image plane quantities are

$$z' = \frac{zf}{z+f},\ z_1' = \frac{z_1 f}{z_1+f},\ \text{and}\ z_2' = \frac{z_2 f}{z_2+f}.$$

Substituting this in the equations for ϵ above and simplifying, we get

$$z_1 - z + z - z_2 = z_1 - z_2 = 2\epsilon f z(z+f)\frac{d}{f^2d^2 - \epsilon^2 z^2}.$$

2.3 A rigid transformation of a point ${}^AP$ can be written as

$$\begin{pmatrix} {}^BP \\ 1 \end{pmatrix} = {}^B_AT \begin{pmatrix} {}^AP \\ 1 \end{pmatrix}$$

where

$${}^B_AT = \begin{pmatrix} {}^B_AR & {}^BO_A \\ 0^T & 1 \end{pmatrix}.$$

For the set of rigid transformations to form a group, we require that a) the matrix product of two rigid transformations to be a rigid transformation matrix, b) the matrix product to be associative, c) there is a unit element and that d) every rigid transformation matrix has an inverse. These properties are all satisfied by ${}^B_AT$. First

$$T_1T_2 = \begin{pmatrix} R_1 & t_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R_2 & t_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_1R_2 & R_1t_2 + t_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_3 & t_3 \\ 0 & 1 \end{pmatrix}$$

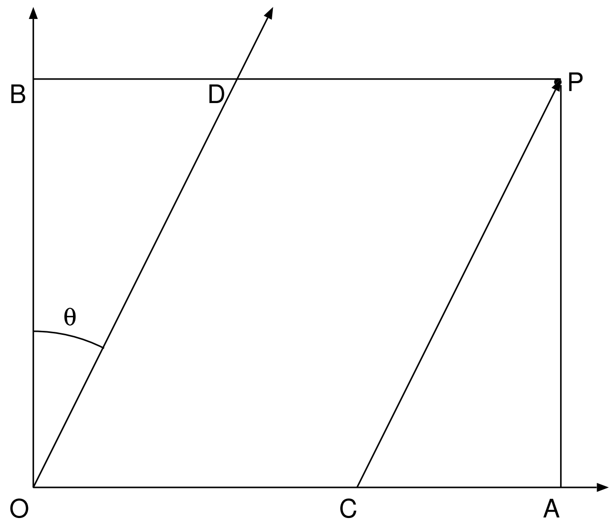
where $R_3 \stackrel{\text{def}}{=} R_1R_2$ and $t_3 \stackrel{\text{def}}{=} R_1t_2 + t_1$. It is trivial to check that $T_1(T_2T_3) = (T_1T_2)T_3$. The identity matrix I acts as an identity element since $TI = IT = T$. Finally, for every T, there exists an inverse matrix $T^{-1} \stackrel{\text{def}}{=} \begin{pmatrix} R^T & -R^Tt \\ 0 & 1 \end{pmatrix}$ such that $TT^{-1} = T^{-1}T = I$. All four properties are therefore verified.

2.7 We have to show that a rigid transformation preserves distances and angles. Assume that we have two points $P_1$and P_2which undergo a rigid transformation. The squared distance between the points before transformation is $||P_1 - P_2||^2 = P_1^TP_1 + P_2^TP_2 - 2P_1^TP_2$. After the transformation, the squared distance is

$$||RP_1 + t - RP_2 - t||^2 = ||RP_1 - RP_2||^2 = P_1^TR^TRP_1 + P_2^TR^TRP_2 - 2P_1^TR^TRP_2 = P_1^TP_1 + P_2^TP_2 - 2P_1^TP_2.$$

Similarly, the transformation preserves angles. Assume that we have two vectors $V_1$and V_2. The inner product between the vectors is $V_1^TV_2$before the transformation. After the transformation, it is $(RV_1)^T(RV_2) = V_1^TR^TRV_2 = V_1^TV_2$. Since the inner product remains the same and since distances are preserved, we know that angles are preserved as well.

2.8 The translation factor is independent of the skew. So this will be ignored. Examine the figure below.



Due to the skew factor, if the old x coordinate is OA, the new x coordinate is OC and if the old y coordinate is OB, the new y coordinate is OD. Since $CA = AP\cot(\theta)$ and $OD = \frac{AP}{\sin(\theta)}$, and given that $OA = \alpha\frac{x}{z}$ and $OB = \beta\frac{y}{z}$, we have

$$OC = \alpha\frac{x}{z} - \beta\cot(\theta)\frac{y}{z}, \text{ and } OD = \beta\frac{1}{\sin(\theta)}\frac{y}{z}.$$

2.15 a) Let $A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{pmatrix}$ and $B = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{pmatrix}$. Then $L = \begin{pmatrix} a_1b_2 - a_2b_1 \\ a_1b_3 - a_3b_1 \\ a_1 - b_1 \\ a_2b_3 - a_3b_2 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$. Since $AB = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$, we have $AB = -(L_3, L_5, L_6)^T$. $OA \times OB = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$. From this, we see that $OA \times OB = (L_4, -L_2, L_1)^T$. Finally, by trivial algebraic manipulation, we can show that $L_1L_6 - L_2L_5 + L_3L_4 = 0$.

b) Since the points A and B are only allowed to drift along the line L we have that a new $A'B' = -(sL_3, sL_5, sL_6)^T$ and that a new $OA' \times OB' = (sL_4, -sL_2, sL_1)^T$. Consequently, the new $L' = sL$.

c) Just substitute using the Plucker formula.

d) Again, just follow the hint and show that $\tilde{M}L \cdot a = 0$ and $\tilde{M}L \cdot b = 0$.

e) We know that A, B and P lie on L. Consequently, the three points and the origin O lie in the same plane. From this, we know that $(OA \times OB) \cdot OP = 0$ and that $(OA \times OB) = OP \times AB$. These two conditions from a set of necessary and sufficient conditions for P to lie on the line joining A and B. Use the four conditions that you obtain from the above and plug in. To see that these conditions are necessary and sufficient, note that $(OA \times OB) \cdot OP = 0$ is satisfied by any four points A, B, O and P that lie in a plane and this can happen without P being collinear with A and B. Similarly, the condition $(OA \times OB) = OP \times AB$ can be satisfied by any four points A, B, O and P that do not necessarily lie in the same plane. However, when one enforces the constraint that A, B, O and P lie in the same plane, then the constraint $(OA \times OB) = OP \times AB$ can only be satisfied by three collinear points A, B and P.

f) If AB lies in plane Π,then $(OA \times OB) \times \mathbf{n} = d\,AB$ where $\mathbf{n} = (a, b, c)^T/\sqrt{a^2 + b^2 + c^2}$ is the normal to the plane. Also, $AB \cdot \mathbf{n} = 0$. These are necessary and sufficient conditions for AB to lie in a plane given by a normal $\mathbf{n}$ and a signed distance d from the origin. Use the four conditions and plug in. To see that these conditions are necessary and sufficient, note that $AB \cdot \mathbf{n} = 0$ is valid for any line lying in the plane and also for any line that does not intersect the plane. The condition $(OA \times OB) \times \mathbf{n} = d\,AB$ can be satisfied by some three points A, B and O and a unit vector n but does not guarantee that $\mathbf{n}$ is perpendicular to AB. When both constraints are enforced, the sole possibility is that AB lies in the plane Π.